

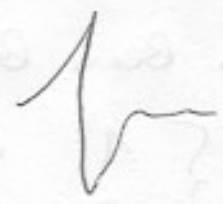
Probabilistic models for networks of spiking neurons

I Intro

①

- neurons communicate by sending sequences of action potentials (spikes)
- fast transmembrane currents of Na^+ / K^+ - ions, stimulated by ion pumps

- spikes are "stereotyped":



always of same slope
(within the same neuron)

Short duration in time: $\approx 1\text{ms}$

followed by refractory period during which the neuron is not able to spike again ($\approx 1\text{ms}$)

=> point process models = microscopic models of what $\text{Alb} \times +$ Pierre has been speaking about

today: discrete time: stochastic gains

monday: continuous time

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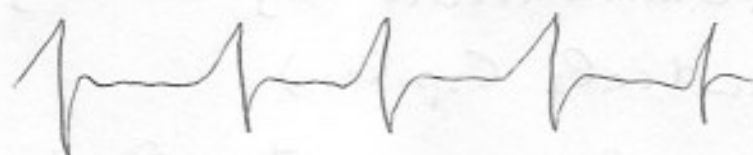
today: discrete time: stochastic gains

monday: continuous time

II The discrete time model

I countable set ($I = \{1, 2, \dots, N\}$): indices of neurons

neuron i :



1 0 0 0 1 0 0 1 0 0 0 1

(time has been binned into intervals of 3 ms)

$$X_t(i) = \begin{cases} 1 & \text{if } i \text{ spikes at time } t \\ 0 & \text{else} \end{cases}$$

$t \in \mathbb{Z}$ index of time window within which we observe the neuron

Observations are raster plots:

i 1 0 1 0 0 0 1 0 1 1

j 0 0 0 1 1 0 1 0 0 1

Stochastic evolution

- neurons update \parallel , given the past:

$$\forall \mathcal{M} \subset I \text{ finite, } \forall a(j) \in \{0, 1\}$$

$$P\left(\bigcap_{i \in \mathcal{M}} \{X_{t+1}(i) = a(i)\} \mid \mathcal{F}_t\right) =$$

$$\prod_{i \in \mathcal{M}} P(X_{t+1}(i) = a(i) \mid \mathcal{F}_t) \quad \begin{matrix} \uparrow \\ \text{past} \leq t \end{matrix}$$

(2)

$$- P(X_{t+1}(i) = 1 | \mathcal{F}_t) = \varphi_i \left(\sum_j \gamma_{j \rightarrow i} \sum_{s=L_t^{i+1}}^t g_i(t-s) X_s(j) \right) \quad (1.1)$$

\uparrow jump rate fct
 \uparrow syn. weight of j on i
 \uparrow leak factor
 \uparrow past spikes received from neuron j at time s , since last spike of i

\int , Lipschitz, \uparrow

$$\varphi_i(V_t(i))$$

where $V_t(i) = \sum_j \gamma_{j \rightarrow i} \sum_{s=L_t^{i+1}}^t g_i(t-s) X_s(j)$

membrane potential of neuron i at time t

Rem 1) proba that i spikes at $t+1$ is a fct φ_i of its membrane potential $V_t(i)$

2) $V_t(i) =$ weighted sum of spikes of presynaptic neurons of i since its last spiking time

$$L_t^i = \sup \{ s \leq t : X_s(i) = 1 \}$$

$$L_t^i = t \Leftrightarrow i \text{ spikes at time } t$$

$$t - L_t^i = \text{age process (} \rightarrow \gamma \text{ (De Vulliamy))}$$

= time elapsed since last spike

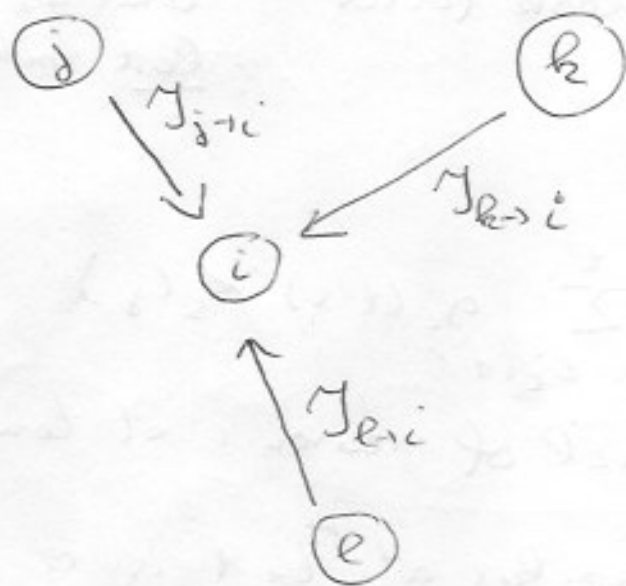
Convention: $\sum_{t+1}^t \dots = 0$, i.e. $V_t(i) = 0$ if

i spikes at time t , that is, $X_t(i) = 1$

At spiking times, the membrane potential is reset to 0.

→ memory structure: (V, L) , non Markovian

"discrete Hawkes"



Conditions

1) Summability of incoming synaptic weights

$$\forall i \quad \sum_j |J_{j \rightarrow i}| < \infty$$

$$2) \quad \forall i \quad g_i = \mathbb{R}_+ + \mathbb{R}_+ \in L^1_{loc}$$

$$3) \quad \forall i \quad \varphi_i \text{ Lipschitz}, \in [0, 1]$$

Ex $I = \{1, 2, 3\}$, $t = 8$ (3)

3:	1	1	0	1	1	0	1	1	
2:	1	0	1	0	0	0	1	0	
1:	1	0	1	1	0	1	0	0	

↓

neuron 3: $L_8^3 = 8 \Rightarrow V_8(3) = 0$
 $\Rightarrow P(X_8(3) = 1 | \hat{I}_8) = \varphi_3(0)$

neuron 2: $L_8^2 = 7 \Rightarrow V_8(2) = \underbrace{\gamma_{3 \rightarrow 2}}_{=1} g_2(0)$

neuron 1: $L_8^1 = 6 \Rightarrow V_8(1) = (\gamma_{3 \rightarrow 1} + \gamma_{2 \rightarrow 1}) g_1(1)$
 $+ \gamma_{3 \rightarrow 1}$

Rem • non-linear Hawkes in discrete time
with a memory structure of VL

→ process with memory!

Markovian description

- possible only for specific choices of leak functions
- either $g_i \equiv \text{const} (= 1)$ no leakage
or $g_i(t) = e^{-\lambda_i t}$

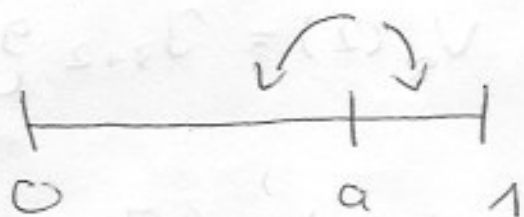
From now on : We work with exp. leak rates

$$g_i(t) = e^{-d_i t}, \quad d_i > 0$$

Simulation also for the evolution of membrane potentials

(let $(u_t(i), i \in I, t \in \mathbb{Z})$ i.i.d., $\sim U[0, 1]$)

$$\forall a \in]0, 1[, \quad P(u_t(i) \leq a) = a$$



$\Rightarrow 1_{[0, a]}$ $(u_t(i)) \sim B(a)$ Bernoulli r.v.

$$V_{t+1}(i) = 1_{[\varphi_i(V_t(i)), 1]} (u_{t+1}(i))$$

no spiking

$$\left\{ e^{-2d_i} V_t(i) + \sum_j \gamma_{j \rightarrow i} 1_{[\varphi_j(V_t(j))]} (u_{t+1}(j)) \right\}$$

(1.2)

indicates if presence of spike for neuron j at $t+1$

Exo • Show that the above definition of $(V_t(i))_{i \in I}$ is equivalent to (1.1) if we

put $X_t(i) = 1_{[0, \varphi_i(V_{t-1}(i))]}(U_t(i))$

• Write down the transition probabilities of $(V_t(i))$ if $I = \{1, 2, 3\}$

III Existence of a stationary version of the process

Stationary means: $\mathcal{L}(X_t(i), i \in I) = \mathcal{L}(X_0(i), i \in I) \forall t$

(or equivalently, $\mathcal{L}(V_t(i), i \in I) = \mathcal{L}(V_0(i), i \in I)$)

Fact: Compact state space

$$V_t(i) = \sum_j \mathcal{T}_{j \rightarrow i} \underbrace{\sum_{s=\tau_{t-1}^i}^t g_i(t-s) X_s(j)}_{\leq \sum_{s=-\infty}^t g_i(t-s) = \frac{1}{1-e^{-d_i}} =: \delta_i}$$

$$\Rightarrow V_t(i) \leq \tau_i \delta_i, \quad \tau_i = \sum_j |\mathcal{T}_{j \rightarrow i}|$$

$\Rightarrow \exists$ invariant measures - but how many?

To prove uniqueness, we have to find some contraction \rightarrow Wasserstein-coupling

Idea Take 2 versions of process V, \tilde{V} , starting from 2 fixed initial conditions v_0, \tilde{v}_0 at time 0

and show that $V_t - \tilde{V}_t \rightarrow 0$ ($t \rightarrow \infty$) in some distance

Coupling Joint construction of V_t, \tilde{V}_t : try to make them jump together (= make the same jumps) as often as possible.

\rightarrow we use the same noise $U_t(i)$ for both processes.

Write $D_t(i) := |V_t(i) - \tilde{V}_t(i)|$

Comment L^1 -distance: convenient for jump processes!

If we could deal with BM, L^2 -distance would be better!

Transitions of $D_t(i)$

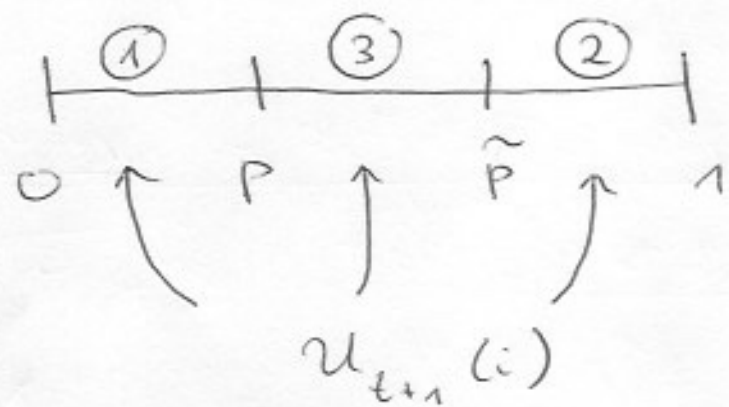
(5)

We know $t \leq t$, and want to see what happens at $t+1$:

$$\text{We know } \varphi_i(V_t(i)) = p_i^0$$

$$p \leq \tilde{p}$$

$$\varphi_i(\tilde{V}_t(i)) = \tilde{p}_i$$



- ① Both spike together 😊
- ② Both do not spike 😊
- ③ $\tilde{V}_{t+1}(i)$ spikes, but not $V_{t+1}(i)$ 😞

Transitions of $D_t(i)$

(6)

1) If $\mathcal{U}_{t+1}(i) = \varphi_i(V_t(i) \wedge \tilde{V}_t(i))$ in both processes, i spikes $\Rightarrow V_{t+1}(i) = \tilde{V}_{t+1}(i) = 0$,
 $D_{t+1}(i) = 0$

2) If $\mathcal{U}_{t+1}(i) = \varphi_i(V_t(i) \vee \tilde{V}_t(i))$ in both processes, i does not spike \Rightarrow

$$D_{t+1}(i) = e^{-d_i} D_t(i) + \sum_j \mathbb{1}_{j \rightarrow i} |X_{t+1}(j) - \tilde{X}_{t+1}(j)|$$

3) If one spikes, but not the other, then

$$D_{t+1}(i) = \max(|V_{t+1}(i)|, |\tilde{V}_{t+1}(i)|) \\ \leq r_i s_i$$

Associated probabilities (given \mathcal{F}_t)

$$2) 1 - \varphi_i(V_t(i) \vee \tilde{V}_t(i)) = 1$$

$$3) |\varphi_i(V_t(i)) - \varphi_i(\tilde{V}_t(i))| = \gamma_i D_t(i)$$

Finally, for 2), we have

$$E[|X_{t+1}(j) - \tilde{X}_{t+1}(j)| | \mathcal{F}_t] = \gamma_j D_t(j)$$

2) + 3) \Rightarrow

$$E D_{t+1}(i) = e^{-d_i} D_t(i) + \sum_j \mathbb{1}_{j \rightarrow i} \gamma_j E D_t(j) \\ + \gamma_i E D_t(i) = r_i s_i$$

$$\text{ou } r_i = \|s_i\|_{L_1}, s_i = \sum_j \mathbb{1}_{j \rightarrow i}$$

So, defining $H_{ii} = e^{-2i} + \gamma_i \tau_i s_i$

$$H_{ij} = \gamma_j |\mathcal{J}_{j \rightarrow i}|, \quad S_t(i) = \mathbb{E} D_t(i)$$

$$\underline{S_{t+1} = H S_t} \rightarrow \text{need } \|H\| < 1$$

Example L_1 -norm: $\|H\|_1 = \sum_{i,j} |H_{ij}| x_j$

$$= \sum_j |x_j| \left(\sum_i |H_{ij}| \right)$$

$$= \sum_j |x_j| \left(\sum_{i \neq j} \gamma_j |\mathcal{J}_{j \rightarrow i}| + e^{-2j} + \gamma_j \tau_j s_j \right)$$

Sufficient condition

$$\sup_j \gamma_j \underbrace{\sum_{k \neq j} |\mathcal{J}_{j \rightarrow k}|}_{\text{what } j \text{ gives}} + e^{-2j} + \gamma_j \tau_j \underbrace{\sum_k |\mathcal{J}_{k \rightarrow j}|}_{\substack{\uparrow \\ \text{what it} \\ \text{receives}}} < 1$$

Prop Under the above condition, $\exists!$ invariant measure, and $S_t \rightarrow 0$ at exp speeds.

Rem In the non-Markovian frame

($g_i(t) \neq e^{-2it}$), a similar proof is possible

by using the distance

$$D_t(i) = 1 - L_t^i \neq \tilde{L}_t^i$$

II Continuous time spiking models I finite (1)

$$P(X_{t_n}(i) = 1 | \mathcal{F}_t) = \varphi_i(V_i(t)) \text{ where}$$

$$V_i(t) = \sum_j \Gamma_{j \rightarrow i} \sum_{s=0}^t \underbrace{g_i(t-s)}_{e^{-\lambda_i(t-s)}} X_s(j) \text{ membrane potential}$$

→ continuous time: probabilities → rates

Time evolution of $V_t(i)$ = PDMP (piecewise deterministic jump process)

- in between successive jumps,

$$dV_i(t) = -\lambda_i V_i(t) dt \quad ; \quad V_i(t) = e^{-\lambda_i t} V_i(0)$$

- neuron i spikes at rate $\varphi_i(V_i(t-)) \geq 0$

This means: $P(i \text{ spikes in } [t, t+h] | \mathcal{F}_t)$

$$= \varphi_i(V_i(t)) \cdot h + o(h), \quad h \rightarrow 0 \quad (2.1)$$

(2.1) will be made rigorous later.

- when spiking, every $j \neq i$ receives an additional amount of potential $\Gamma_{i \rightarrow j}$:

$$V_j(t) = V_j(t-) + \Gamma_{i \rightarrow j}$$

Hawkes-frame

with exponential memory-functions.

Equivalent description I finite

$$V(t) = (V_i(t), i \in I)$$

generator (recall the talk of Pierre Gabriel):

$$P_t f(v) = E_v [f(V_t)] \quad \text{f nice test function} \quad \textcircled{*}$$

$$\text{Then } Af(v) = \lim_{t \downarrow 0} \frac{P_t f(v) - f(v)}{t}$$

$$= \sum_{i \in I} \frac{\partial f}{\partial v_i} (-\lambda_i v_i) + \sum_{i \in I} \varphi_i(v_i) [f(v + \Delta_i) - f(v)]$$

$$\text{where } (\Delta_i)_j = \begin{cases} \gamma_{ij}, & j \neq i \\ \gamma_{ii}, & j = i \end{cases}$$

↑
each neuron spikes,
at rate $\varphi_i(v_i)$,
depending on its
potential value.

* if I is finite:
f depends only on
finite # of coordinates

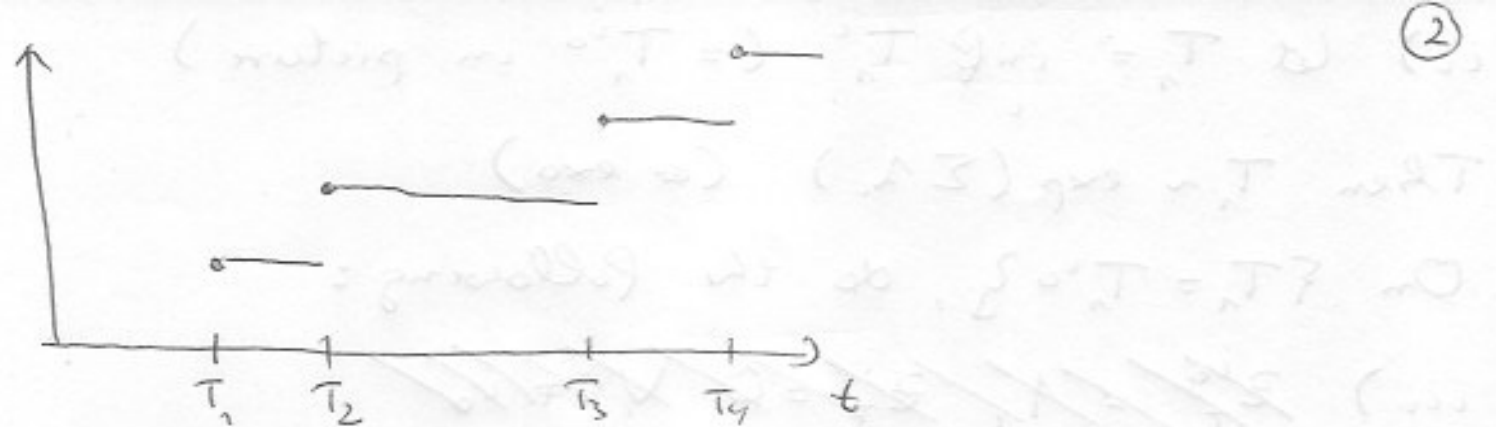
Jump rates

• jump rates are time varying parameters of
extended exponential r.v.

• $\tau \sim \exp(\lambda)$ - proba density $\lambda e^{-\lambda t} \mathbb{1}_{\mathbb{R}_+}(t)$
- $P(\tau > t) = e^{-\lambda t} \quad \forall t \geq 0$

• N_t Poisson process of rate λ

$(\tau_n)_{n \geq 1}$ i.i.d., $\exp(\lambda)$, $T_n = \tau_1 + \dots + \tau_n$



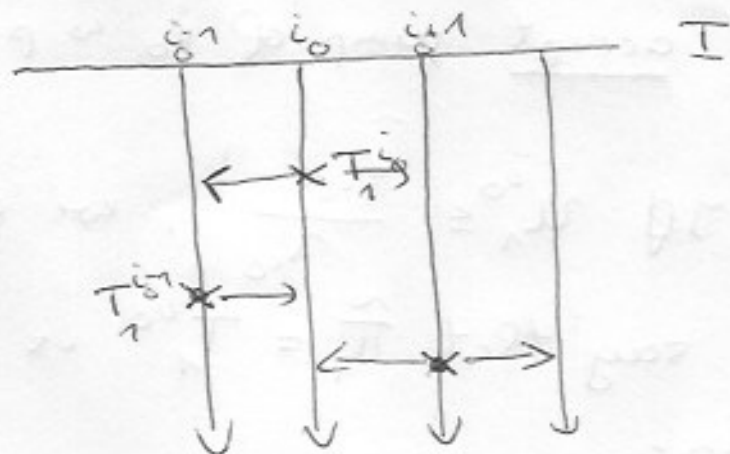
- $N_t = \max \{n : T_n \leq t\} = \# \text{ events seen before } t$
- 1. jump arrives at T_1 , $E T_1 = \frac{1}{\lambda} \Rightarrow$ the higher λ , the more events we see
- $dN_t = 1$ at $t = T_n$ for some n
- $N_t \sim \text{Poisson}(\lambda t) \forall t, E N_t = \text{Var } N_t = \lambda t$

Back to process (2.1)

MF $\varphi_i(v) \equiv \lambda_i \forall v, v_i$, a possible simulation algo

for (2.1) is:

- i) Take \parallel Poisson processes $N_t^i, i \in I$, each of rate λ_i



$(T_n^i) =$ successive jumps of N_t^i ; of neuron i
"spikes"

ii) let $T_1 = \inf_i T_1^i$ ($= T_1^{i_0}$ in picture)

Then $T_1 \sim \exp(\sum \lambda_i)$ (\leftarrow exo)

On $\{T_1 = T_1^{i_0}\}$, do the following:

iii) ~~$z_{T_1}^{i_0} = 1, z_{T_1}^i = 0 \forall i \neq i_0$~~

$$\forall t < T_1, \forall i \quad V_i(t) = e^{-\lambda_i t} V_i(0)$$

$$(2.2) \quad \boxed{V_i(T_1) = V_i(T_1^-) + J_{i_0 \rightarrow i}}$$

Iterate!

If the jump rates are not constant:

$$\textcircled{A} \quad \|\varphi_i\|_\infty \leq \lambda_i \quad \forall i$$

Let N_t^i be as before, write $(\tilde{T}_n^i)_{n \geq 1}$ for the associated jumps, let $u_n^{i_0} \perp \perp U[0,1]$

$\tilde{T}_1 = \inf \tilde{T}_1^i$. On $\{\tilde{T}_1 = \tilde{T}_1^{i_0}\}$:

accept jump of i_0 w.p. $\frac{\varphi_{i_0}(V_{i_0}(\tilde{T}_1^-))}{\lambda_{i_0}}$

Means: If $u_1^{i_0} \leq \frac{\varphi_{i_0}(-)}{\lambda_{i_0}}$, we accept \tilde{T}_1

and we say that $\tilde{T}_1 = T_1^{i_0}$ is a spike of neuron i_0 .

Then we perform iii) as above. And:

$$z_{T_1}^{i_0} = 1, z_t^{i_0} = 0 \quad \forall t < T_1, \quad z_{T_1}^j = 0 \quad \forall j \neq i_0$$

Otherwise, we iterate:

$$\tilde{T}_2 = \inf \{ t > \tilde{T}_1 : \Delta N_c^j = 1 \text{ for some } j \},$$

and on $\{ \tilde{T}_2 = \tilde{T}_R^i \}$ for some i and R , we do the same.

$T_1 := \inf \{ \tilde{T}_n : \text{acceptedly } \tilde{T}_n \text{ is the 1. jump of } V(t) \}$

$$\text{Prop } P(T_1 > t) = e^{-\sum_{i \in I} \int_0^t \varphi_i(e^{-\lambda_i s} v_i(0)) ds}$$

$$E Z_t^i = E \int_0^t \varphi_i(V_i(s)) ds$$

(B) φ_i only Lipschitz, $\varphi_i: \mathbb{R} \rightarrow \mathbb{R}_+$, not bounded

Start at $t=0$ with $v_i(0), i \in I$

$$\lambda_i := \sup \{ \varphi_i(e^{-\lambda_i t} v_i(0)), t \geq 0 \}$$

= max. possible jump rate of i before 1. spike

Update the $(\lambda_i)_{i \in I}$ after each jump, using the new potential values $V_i(T_1) \leq v_i(0)$

Prop No explosion! (Explain) $E[Z_t^i] < \infty$

Proof $\tau_K = \inf \{ t > 0 : |V_i(t)| \geq K \text{ for some } i \}$

Up to time τ_K , all rates are bounded

$$\lambda_i := \sup \{ \varphi_i(v) : |v| \leq K \}$$

Let N_c^i be the associated PP

$$V_i(t \wedge \tau_K) = V_i(0) - \lambda_i \int_0^{t \wedge \tau_K} V_i(s) ds$$

$$+ \sum_j \Gamma_{j \rightarrow i} \int_0^{t \wedge \tau_K} \underbrace{1 \cdot \{ \cancel{u_{s-}^j} \}}_{\varphi_j(V_i(s-)) / \lambda_j} dN_s^j$$

$$\begin{aligned} E(\dots) &= E \int_0^{t \wedge \tau_K} \frac{\varphi_j(V_i(s))}{\lambda_j} \lambda_j ds \\ &= \int_0^t E[\varphi_j(V_i(s \wedge \tau_K))] ds \end{aligned}$$

$$\Rightarrow E[V_i(t \wedge \tau_K(i))] \leq V_i(0) + \sum_j |\Gamma_{j \rightarrow i}| \int_0^t E[\varphi_j(V_i(s \wedge \tau_K))] ds$$

Use: $\varphi_j(v) \leq \varphi_j(0) + \gamma_j v + \text{Grönwall}$ \square

$$\text{ou: } + \sum_j \Gamma_{j \rightarrow i} z_{t \wedge \tau_K}^j$$

$$E(\dots) = E \int_0^{t \wedge \tau_K} \varphi_j(V_i(s \wedge \tau_K)) ds$$

Mean field frame

(4)

$$I = \{1, \dots, N\}, \varphi_i \equiv \varphi, \gamma_{i,j} = \frac{1}{N} \forall i, j, \tau_i = \tau$$

=> each neuron i spikes at the same rate

$\varphi(V^N(t))$, but is driven by its own PP N_t^i \perp its own u_i^c

$$- dV^N(t) = -\lambda V^N(t) dt + \frac{1}{N} \sum_{j=1}^N dZ_t^{N,j}$$

where $Z_t^{N,j}$ is the spike counting process of neuron j
= # {spikes of j in $[0, t]$ }
with intensity $\varphi(V^N(t))$: accept jumps of N^j with $\frac{\varphi(\cdot)}{\lambda}$

Heuristics: $\frac{1}{N} \sum_{j=1}^N dZ_t^{N,j} \rightarrow dE \bar{Z}_t$

$$\alpha: \frac{1}{N} \sum_{j=1}^N Z_t^{N,j} \rightarrow E \bar{Z}_t$$

where \bar{Z}_t is the spiking counting process of a typical neuron in the limit

Since $dE Z_t^{N,j} = E[\varphi(V^N(t))] dt$

$$dE \bar{Z}_t = E[\varphi(\bar{V}(t))] dt$$

$\bar{V}(t)$ = membrane potential of a typical neuron in the limit. Should solve:

$$(2.2) \quad \boxed{d\bar{V}(t) = -\lambda \bar{V}(t) dt + E[\varphi(\bar{V}(t))] dt}$$

← explain with \bar{V}
PP $dEN_t^i = \lambda dt$

MF in continuous time

(1)

N neurons, T_n^i successive spiking times of neuron i , $i \in N_c$ PP(λ), $(u_n^i)_{n \geq 1}$ i.i.d. $u \in [0, 1]$

$$\bullet z_t^{N,i} = \sum_{n=1}^{\infty} 1_{T_n^i \leq t} = \sum_{n=1}^{\infty} \delta_{T_n^i}([0, t])$$

$$\bullet E z_t^{N,i} = \int_0^t E[\varphi(V^N(s))] ds$$

$$\bullet dV^N(t) = -\lambda V^N(t) dt + \frac{1}{N} \sum_{j=1}^N dz_t^{N,j}$$

$\bullet z^{N,j}$ is a thinning of N^j PP($\|\varphi\|_{\infty}$) - PP(λ)
(accept each jump $[t]$ of N^j w.p. $\frac{\varphi(V^N(t))}{\lambda}$)

$N^j \perp\!\!\!\perp$

limit system as system of neurons,

\bar{T}_n^i successive spiking times of neuron i

$$\bar{z}_t^i = \sum_{n=1}^{\infty} 1_{\bar{T}_n^i \leq t}$$

$$E \bar{z}_t^i = \int_0^t \varphi(\bar{V}(s)) ds$$

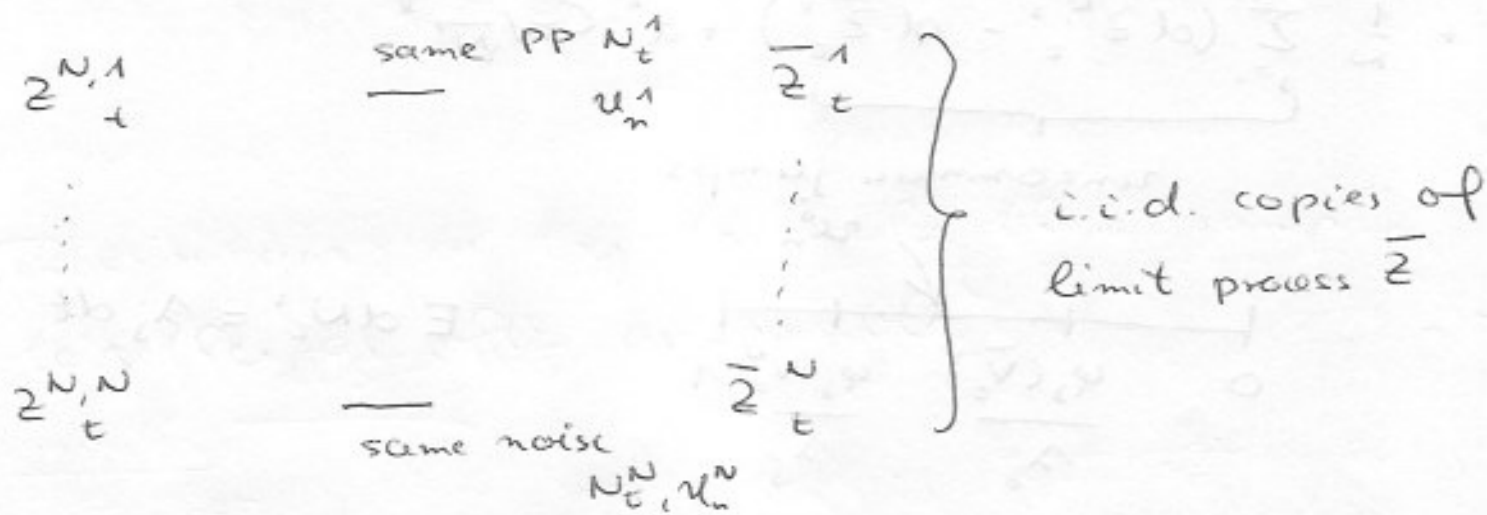
$$d\bar{V}(t) = -\lambda \bar{V}(t) dt + \underbrace{\varphi(\bar{V}(t)) dt}_{\text{accumulated gain coming from the joint action of the other neurons.}}$$

Semitrivial coupling Construct \bar{z}^i using

SAHE PP(λ) N^i AND SAHE u_n^i
as those constructing $z^{N,i} \Rightarrow \bar{z}^i, z^{N,i}$ have
SAHE candidate jump times + some acceptance var.

This means: we try to make them jump together as often as possible.

HYP: (for simplicity) $\|\varphi\|_\infty \leq \lambda$. Let (N_t^i) (5)
 i.i.d. PP(λ)



Same noise means: We use the same candidate jump times for $z^{N,i}$ and \bar{z}^i , for all i .

AND we use the same uniform random variables to accept or not the jumps

\Rightarrow differences can only be due to uncommon jumps as in the 1. lecture.

Proof of convergence

$$\text{Rewrite: } dV^N(t) = -\lambda V^N(t) dt + \frac{1}{N} \sum_{j=1}^N dZ_t^{N,j}$$

$$d\bar{V}(t) = -\lambda \bar{V}(t) dt + \frac{1}{N} \sum_{j=1}^N d\bar{Z}_t^j$$

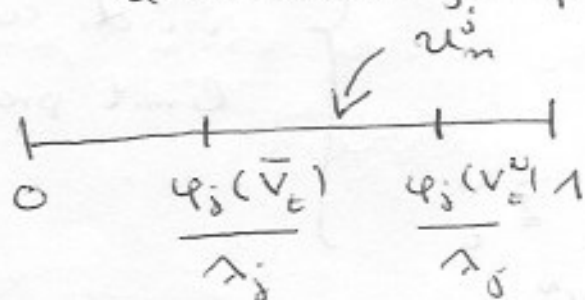
$$\Rightarrow \frac{1}{N} \sum_{j=1}^N \underbrace{\left[d\bar{Z}_t^j - \varphi(\bar{V}(t)) \right]}_{\text{centered i.i.d. PP, } \varphi(\cdot) \text{ bounded}} dt$$

\Rightarrow this is a variance term; $E(-) \leq C_T / \sqrt{N}$
 $\forall t \in [0, T]$

$$\Rightarrow d(V^N(t) - \bar{V}(t)) = -2(V^N(t) - \bar{V}(t))dt$$

$$+ \frac{1}{N} \underbrace{\sum_{j=1}^N (dZ_t^{N,j} - d\bar{Z}_t^j)}_{\text{uncommon jumps}} + d'' C_T / \sqrt{N}''$$

uncommon jumps



$$E dN_t^j = \lambda_j dt$$

$$E(\text{uncommon jumps}) \leq \|\varphi\|_{L^p} E[|V^N(t) - \bar{V}(t)|]$$

Gron wall \Rightarrow

Prop $\forall t \leq T$, if φ Lipschitz

$$\bullet E[|V^N(t) - \bar{V}(t)|] \leq C_T / \sqrt{N}$$

$$\bullet E\left[\sup_{t \leq T} |Z_t^{N,R} - \bar{Z}_t^R|\right] \leq C_T / \sqrt{N}, \forall R$$

$$\bullet \forall R, \left((Z_t^{N,i})_{t \geq 0}, 1 \leq i \leq R \right) \xrightarrow{L} P^{\otimes R}$$

where $P = L((\bar{Z}_t)_{t \geq 0})$ is the law of the limit process.

Rem The above proof is the "Markovian" version of a proof that works for general Hawkes processes given by Delattre, Hoffmann, Fournier AAP 2016

Rem In our original model, (5')
 the membrane potential of spiking neuron is reset
 to 0. We did not consider this here.

If we add this feature, we obtain a limit
 process $\bar{V}(t)$ spiking at rate $\varphi(\bar{V}(t-))$
 and reset to 0 when spiking

+ evolution $d\bar{V}(t) = -\lambda \bar{V}(t) dt + E[\varphi(\bar{V}(t)) dt]$

(same as before, but we cannot take away the
 expectation! since $\bar{V}(t)$ remains random)

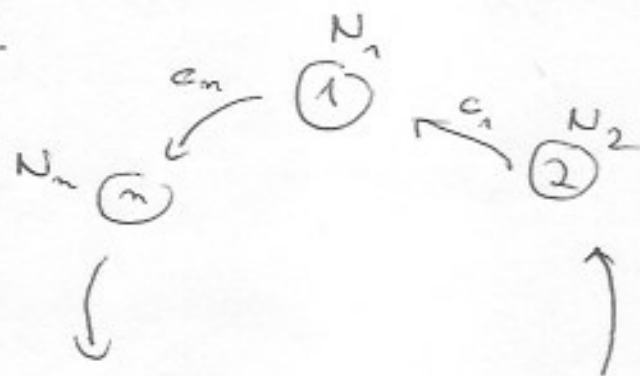
The law $g(t, v)$ of $\bar{V}(t)$ satisfies nonlinear PDE

$$\frac{d}{dt} \int_0^{\infty} \phi(x) g(t, dv) = \int_0^{\infty} [\phi(0) - \phi(v)] \varphi(v) g(t, dv) + \int_0^{\infty} \phi'(v) [p_t - \lambda v] g(t, dv)$$

$$p_t = \int_0^{\infty} \varphi(v) g(t, dv)$$

$$\text{or: } \begin{cases} \partial_t g(t, v) = (\lambda v - p_t) \partial_v g(t, v) + (\lambda - \varphi(v)) g(t, v) \\ g(t, 0) \equiv 1 \end{cases}$$

Multi-pop



• $N = N_1 + \dots + N_n \rightarrow \infty$

• all neurons within one pop. k are interchangeable: they have the same rate

$$\varphi_k(V_k^N(t))$$

$$dV_k^N(t) = -\lambda_k V_k^N(t) dt + c_k \frac{1}{N_{k+1}} \sum_{j=1}^{N_{k+1}} dZ_{k+1,j}^N(t)$$

⑥
thinning of i.i.d. underlying PP ($\mathbb{P}(V_{k+1}, V_{k+2})$)

Limit System $k=1, \dots, n$

$$\bar{V}_k'(t) = -\lambda_k \bar{V}_k(t) + c_k \varphi_{k+1}(\bar{V}_{k+1}(t))$$

Monotone cyclic feed back

$$S := \prod_{k=1}^n c_k \quad \text{total feedback}$$

TR If $S < 0$ and φ_k bounded + analytical

then:

1) $\exists!$ equilibrium $v^* = (v_1^*, \dots, v_n^*)$

2) If its unstable $\Rightarrow \exists$ finite # of non-constant periodic orbits

Any ω -limit set must be one of these orbits

(Mallet-Paret + Smith '90)