Global L^1 theory and regularity for the 3D nonlinear Wigner–Poisson–Fokker–Planck system*

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Abstract

A global existence, uniqueness and regularity theorem is proved for the simplest Markovian Wigner–Poisson–Fokker–Planck model incorporating friction and dissipation mechanisms. The proof relies on Green function and energy estimates under mild formulation, making essential use of the Husimi function and the elliptic regularization of the Fokker–Planck operator.

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1 Introduction and main result

The modeling of quantum diffusion is one of the fields of mathematical interest in quantum mechanics that is not completely well understood at present. Some remarkable works aiming to an earlier analysis of diffusive corrections in models arising from quantum kinetics are due to Caldeira and Leggett [4], Diósi [8, 9] and Diósi *et al.* [10]. The proper framework of such sort of diffusive models is that of open quantum systems, i.e. an ensemble of electrons interacting with a heat bath (an infinite set of harmonic oscillators in thermodynamic equilibrium) that can exchange matter (conserved particles) with their environment (see [4], [7], [11], [6]).

The quantum Wigner–Fokker–Planck equation reads

$$\frac{\partial W}{\partial t} + (\xi \cdot \nabla_x) W + \Theta[V]W = \frac{D_{pp}}{m^2}\Delta_\xi W + 2\lambda \mathrm{div}_\xi(\xi W) + 2\frac{D_{pq}}{m}\mathrm{div}_x(\nabla_\xi W) + D_{qq}\Delta_x W\,, \quad (1)$$

where W is the (quasi)–probability distribution function, D_{pp}, D_{pq}, D_{qq}, m and λ are physical constants and $\Theta[V]W$ is the (quadratic) nonlinear term associated with the 3D Hartree self–consistent potential (cf. (7) below). In a recent paper [1], the well–posedness of so–called Wigner–Poisson–Fokker–Planck (WPFP) system in the simplest Markovian approach for the (high temperature) frictionless case ($\lambda = 0$) is studied. This is a quantum–kinetic model (in the Wigner representation) with Fokker–Planck dissipation mechanism only in the ξ–direction (that is, $D_{pq} = D_{qq} = 0$):

$$\frac{\partial W}{\partial t} + (\xi \cdot \nabla_x) W + \Theta[V]W = \frac{D_{pp}}{m^2}\Delta_\xi W, \quad x, \xi \in \mathbb{R}^3,\ t > 0\,. \quad (2)$$

The Lindblad form [13] of this kinetic operator at the density matrix level implies that the problem is mathematically consistent, in the sense that the equation preserves the positivity of the initial density matrix. The problems of local existence, uniqueness, stability, regularity and long–time behaviour (in the case of global solutions) of the Wigner function mild solutions of (2) are also tackled in [1]. In [6], the authors make a mathematically rigorous derivation of the frictionless Fokker–Planck equation from the Caldeira–Leggett model introduced in [4]. Furthermore, they investigate other Fokker–Planck–type equations obtained from the Caldeira–Leggett Hamiltonian through different diffusion mechanisms and scalings (fixed temperature and long–time limit), especially a heat equation with a friction term for the radial process in phase space. Also, the rate of time decay of solutions to the viscous hydrodynamic model (i.e. the moment equations for the charge density and the current coupled to the Poisson equation for the electric potential) associated with the 1D WPFP equation is studied in [12] via the entropy dissipation method.

This paper is devoted to prove the existence of global mild solutions (i.e. solutions of the WPFP equation written in an equivalent integral form, defined in $[0,\infty)$) to the most general physically relevant class of WPFP models (we only set $D_{pq}=0$). Indeed, we are concerned with the analysis of the following initial value problem:

$$\frac{\partial W}{\partial t} + (\xi\cdot\nabla_x)W + \Theta[V]W = \frac{D_{pp}}{m^2}\Delta_\xi W + 2\lambda \mathrm{div}_\xi(\xi W) + D_{qq}\Delta_x W \tag{3}$$

$$W(x,\xi,0) = W_0(x,\xi)\,, \tag{4}$$

coupled to the Poisson equation for the determination of the self–consistent electrostatic potential:

$$V(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3_y} \frac{n(y,t)}{|x-y|}\,dy\,, \tag{5}$$

with

$$n(x,t) = \int_{\mathbb{R}^3_\xi} W(x,\xi,t)\,d\xi\,. \tag{6}$$

Here, $\Theta[V]$ stands for the pseudo–differential operator

$$\Theta[V]W(x,\xi,t) = \frac{i}{(2\pi)^3}\int_{\mathbb{R}^3_\eta}\int_{\mathbb{R}^3_{\xi'}} \frac{V(x+\frac{\hbar}{2m}\eta,t) - V(x-\frac{\hbar}{2m}\eta,t)}{\hbar}$$

$$\times W(x,\xi',t)e^{-i(\xi-\xi')\cdot\eta}\,d\xi'\,d\eta\,, \tag{7}$$

with $\hbar$ denoting the reduced Planck constant and m the effective mass of the particles, while λ, D_{pp}, D_{qq} are positive constants related to the interactions between the particles and the reservoir (cf. [8]):

$$\lambda = \frac{\eta}{2m}\,, \quad D_{pp} = \eta k_B T\,, \quad D_{qq} = \frac{\eta\hbar^2}{12m^2 k_B T}\,, \tag{8}$$

where $\eta > 0$ is the coupling (damping) constant of the bath, k_B the Boltzmann constant and T the temperature of the bath. Also,

$$Q = \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_\xi} W(x,\xi,t)\,d\xi\,dx$$

is the total charge of the system, which is preserved along the evolution. This equation is the simplest systematic Markovian approximation taking friction and dissipation effects into account, such that the corresponding master equation for the density matrix of the particle ensemble still belongs to the Lindblad class (as shown in [9]), ensuring preservation of positivity for all initial conditions and for all times. Indeed, if the elliptic term involving $\Delta_x W$ is removed from Eq. (3), then the remaining Fokker–Planck collision kernel (accounting only for friction and ξ–diffusion effects) prevents the equation from belonging to the Lindblad family. Thus, in this case the problem would be neither mathematically consistent nor meaningful in a physical context.

The WPFP equation (3) stems from the following evolution model (see [1]) for the density matrix function $\rho(x,y,t) \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_y)$:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}(H_x - H_y)\rho - \lambda(x-y)\cdot(\nabla_x - \nabla_y)\rho$$
$$+ \left(D_{qq}|\nabla_x + \nabla_y|^2 - \frac{D_{pp}}{\hbar^2}|x-y|^2\right)\rho\,,$$

where H_x and H_y are copies of the electron Hamiltonian

$$H_z = -\frac{\hbar^2}{2m}\Delta_z + V(z,t)$$

acting on the variables x and y, respectively. Indeed, taking into account that the Wigner function of the electron ensemble $W : \mathbb{R}^3_x \times \mathbb{R}^3_\xi \times [0,\infty) \to \mathbb{R}$ is defined by

$$W(x,\xi,t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3_\eta} \rho\Big(x + \frac{\hbar}{2m}\eta, x - \frac{\hbar}{2m}\eta, t\Big) e^{-i\xi\cdot\eta}\, d\eta\,,$$

one can easily deduce that the evolution law for $W(x,\xi,t)$ is described by Eq. (3). The positivity of the density matrix operator

$$[R(t)f](x) = \int_{\mathbb{R}^3_y} f(y)\rho(x,y,t)\, dy \in L^2(\mathbb{R}^3)$$

(guaranteed by the Lindblad condition) implies that the Husimi transform, defined by the following convolution of the Wigner function with a Gaussian kernel

$$W^H(x,\xi,t) = W(x,\xi,t) *_{x,\xi} \left(\frac{m}{\hbar\pi}\right)^3 \exp\left\{-\frac{m}{\hbar}\Big(|x|^2 + |\xi|^2\Big)\right\}, \qquad (9)$$

is pointwise nonnegative on $\mathbb{R}^3_x \times \mathbb{R}^3_\xi$. Also, the Lindblad condition and a nonvanishing friction ($\lambda > 0$) imply that the Fokker–Planck operator is uniformly elliptic in $\mathbb{R}^3_x \times \mathbb{R}^3_\xi$.

Contrary to the common techniques leading to the global existence, regularity and asymptotic behaviour of solutions to classical (Vlasov)–Fokker–Planck systems, our techniques overcome the explicit use of S_p norms (see [5, 2, 15] for example) to control the position density. Actually, our proof does not require more regularity than $L^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi) \cap L^1(\mathbb{R}^3_\xi; L^2(\mathbb{R}^3_x))$ and the control of the kinetic energy for the initial Wigner function. We also remark that the natural space where the Wigner function lives is $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$, as sheds from the original density matrix formulation in the widest context of Wigner problems. Actually, by formally multiplying Eq. (3) by W and integrating against x and ξ we have

$$\|W(t)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} \le \|W_0\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}\, e^{6\lambda t}\,.$$

However, the presence of a regularizing Fokker–Planck kernel in the model under study allows us to develop a (surprising) L^1 theory for the WPFP equation and exploit the smoothing properties of the Fokker–Planck operator to get regular solutions. It is significant the fact that no maximum principle is available for equations of Wigner type, so that in general the Wigner function changes sign even if we start from positive initial data. This is why the Husimi function (9) together with the elliptic regularization in the x–variable will play an essential (although seemingly hidden) role in our analysis.

We prove the following global–in–time existence result:

Theorem 1.1 *Let* $W_0 \in L^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi) \cap L^1(\mathbb{R}^3_\xi; L^2(\mathbb{R}^3_x))$ *be such that* $\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_\xi} |\xi|^2 W_0(x,\xi)\, d\xi\, dx < \infty$. *Then, the Wigner–Poisson–Fokker–Planck equation (3)–(8) admits a unique global mild solution*

$$W \in C([0,\infty); L^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)) \cap C([0,\infty); L^1(\mathbb{R}^3_\xi; L^2(\mathbb{R}^3_x)))\,.$$

Moreover,

$$W \in C((0,\infty); W^{1,1} \cap W^{1,\infty}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))\,.$$

Also, the charge density (6) *and the electric potential* (5) *satisfy the following Hölder–regularity properties: for all* $t > 0$,

$$n(\cdot,t) \in C^{0,\alpha}(\mathbb{R}^3_x) \text{ with } 0 < \alpha < \frac{1}{2}\,, \quad V(\cdot,t) \in C^{1,\beta}(\mathbb{R}^3_x) \text{ with } 0 < \beta < \frac{1}{3}\,.$$

The paper is structured as follows: in Section 2 we construct the fundamental solution of the linear kinetic Fokker–Planck operator and establish its main properties. Section 3 concerns the local existence and uniqueness of solutions to the 3D WPFP system with nonvanishing friction. In Section 4 we show some regularization effects of the Fokker–Planck kernel on the Wigner function, the charge density and the electric potential. Finally, Section 5 is devoted to prove that the solution found in Section 3 exists globally in time.

2 On the fundamental solution

This section is devoted to the description of the fundamental solution of the quantum Fokker–Planck model (3)–(7) and the derivation of some of its main properties. The Green function G associated with the linear kinetic Wigner–Fokker–Planck equation under study is the fundamental solution of

$$L[W] := \frac{\partial W}{\partial t} + (\xi \cdot \nabla_x) W - \frac{D_{pp}}{m^2} \Delta_\xi W - 2\lambda \mathrm{div}_\xi(\xi W) - D_{qq} \Delta_x W = 0\,. \quad (10)$$

Lemma 2.1 *The fundamental solution of the linear operator (10) is given by*

$$G(x, \xi, z, v, t) = G_0\Big(x - z - \Big(\frac{1 - e^{-2\lambda t}}{2\lambda}\Big) v, \xi - e^{-2\lambda t} v, t\Big), \quad (11)$$

where

$$G_0(x, \xi, t) = d(t) \exp\Big\{-a(t)|x|^2 + b(t)(x \cdot \xi) - c(t)|\xi|^2\Big\} \quad (12)$$

with coefficients

$$a(t) = \frac{m^2 \lambda^3}{D_{pp}} \frac{(1 - e^{-4\lambda t})}{D(t)}, \quad (13)$$

$$b(t) = \frac{m^2 \lambda^2}{D_{pp}} \frac{(1 - e^{-2\lambda t})^2}{D(t)}, \quad (14)$$

$$c(t) = \frac{m^2 \lambda}{4 D_{pp}} \frac{\Big(4\lambda t \left(1 + 4\lambda^2 m^2 \frac{D_{qq}}{D_{pp}}\right) - (1 - e^{-2\lambda t})(3 - e^{-2\lambda t})\Big)}{D(t)}, \quad (15)$$

$$d(t) = \left(\frac{\sqrt{4a(t)c(t) - b(t)^2}}{2\pi}\right)^3 = \left(\frac{m^2 \lambda^2}{\pi D_{pp} \sqrt{D(t)}}\right)^3, \quad (16)$$

and where

$$D(t) = \lambda\Big(1 + 4\lambda^2 m^2 \frac{D_{qq}}{D_{pp}}\Big) t (1 - e^{-4\lambda t}) - (1 - e^{-2\lambda t})^2\,. \quad (17)$$

Notice that $D(t)$ and $4a(t)c(t) - b(t)^2$ are positive functions for all positive times.

The proof is based on the Fourier transformation of Eq. (10) with respect to the (x, ξ)–variables and then integration of the resulting linear first order hyperbolic problem for $\hat{G}(y, \eta, t)$

$$\frac{\partial \hat{G}}{\partial t} - (y \cdot \nabla_\eta)\hat{G} + 2\lambda(\eta \cdot \nabla_\eta)\hat{G} + \frac{D_{pp}}{m^2}\eta^2\hat{G} + D_{qq}y^2\hat{G} = 0\,,$$
$$\hat{G}(y, \eta, 0) = 1\,,$$

along the characteristics $\eta \mapsto e^{2\lambda t}\eta + \frac{1}{2\lambda}(1 - e^{2\lambda t})y$, where we denoted

$$\hat{G}(y, \eta, t) = \mathcal{F}_{x \mapsto y, \xi \mapsto \eta} G(y, \eta, t) = \iint_{\mathbb{R}^3_x \mathbb{R}^3_\xi} G(x, \xi, t) e^{i(x \cdot y + \xi \cdot \eta)}\, d\xi\, dx\,.$$

From now on, when there is no possible confusion we shall denote

$$L^p = L^p(\mathbb{R}^3_x \times \mathbb{R}^3_\xi), \quad L^{q,p} = L^q(\mathbb{R}^3_\xi; L^p(\mathbb{R}^3_x))\,.$$

In the following result we list some of the properties of G that will be useful in the sequel to deal with mild solutions of the system (3)–(7). We have

Lemma 2.2 *The fundamental solution G of the linear kinetic Fokker–Planck equation (10), given by formulae (11)–(17), satisfies the following properties for any $t \geq 0$:*

(i) $\int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_\xi} G(x, \xi, z, v, t)\, d\xi\, dx = 1$ *for all* $z, v \in \mathbb{R}^3$.

(ii) $\|G_0(t)\|_{L^{q,p}} \leq C(q, p)\, a(t)^{\frac{3}{2}\left(\frac{1}{q} - \frac{1}{p}\right)}\, d(t)^{1 - \frac{1}{q}}$ *for all* $1 \leq q, p < \infty$.

(iii) For all $1 \leq q \leq p < \infty$*, we have*

$$\|\nabla_{(x,\xi)} G_0(t)\|_{L^p(\mathbb{R}^3_x; L^q(\mathbb{R}^3_\xi))} \leq C(p, q)\Big[\Big(2a(t) + b(t)\Big) d(t)^{\frac{2}{3} - \frac{1}{p}}\, c(t)^{\frac{1}{2}\left(1 + \frac{3}{p} - \frac{3}{q}\right)} + \Big(2c(t) + b(t)\Big) d(t)^{\frac{2}{3} - \frac{1}{q}}\, a(t)^{\frac{1}{2}\left(1 + \frac{3}{q} - \frac{3}{p}\right)}\Big]\,.$$

In particular, for $q = p$ we have

$$\|\nabla_{(x,\xi)} G_0(t)\|_{L^p} \le C(p) d(t)^{\frac{2}{3}-\frac{1}{p}} \Big[\Big(2a(t)+b(t)\Big)\sqrt{c(t)} + \Big(2c(t)+b(t)\Big)\sqrt{a(t)} \Big].$$

Proof. The proof follows from elementary computations. We just make a few remarks on the proof of (iii). First we observe that

$$|\nabla_{(x,\xi)} G_0(x,\xi,t)| \le \Big[\Big(2a(t)+b(t)\Big)|x| + \Big(2c(t)+b(t)\Big)|\xi| \Big] G_0(x,\xi,t)\,.$$

Then, we can estimate

$$\begin{aligned} \|\nabla_{(x,\xi)} G_0(t)\|_{L^p(\mathbb{R}^3_x;L^q(\mathbb{R}^3_\xi))} \le\ & \Big(2a(t)+b(t)\Big)\| |x| G_0(t)\|_{L^p(\mathbb{R}^3_x;L^q(\mathbb{R}^3_\xi))} \\ & + \Big(2c(t)+b(t)\Big)\| |\xi| G_0(t)\|_{L^{q,p}}\,, \end{aligned}$$

where we have used Minkowski's inequality to reverse the order of the norms acting on $|\xi| G_0$. ■

3 Existence of local–in–time mild solutions

In this section we prove the existence and uniqueness of local–in–time mild solutions to the 3D WPFP system (3)–(8) by application of a fixed–point argument of contractive type. Local existence was also dealt with in [1] for the simplest frictionless Wigner–Fokker–Planck model (2). By a mild solution $W(x,\xi,t)$ of the WPFP system (3)–(4) we understand that satisfying the following integral equation:

$$\begin{aligned} W(x,\xi,t) = & \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_z} G(x,\xi,z,v,t) W_0(z,v)\, dz\, dv \\ & - \int_0^t \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_z} G(x,\xi,z,v,t-s)\Theta[V]W(z,v,s)\, dz\, dv\, ds\,. \end{aligned} \tag{18}$$

Clearly, from the concept of mild solution we may consider W to be split into two parts: the linear part, only depending on the initial data W_0, and the nonlinear part depending upon the potential V through the pseudo–differential operator $\Theta[V]W$. Also, we observe that the first term in the above decomposition actually solves the linear Wigner–Fokker–Planck problem (10) with initial data W_0.

Henceforth in the paper the following identity for the nonlinear term constitutes a crucial ingredient (see [1]):

$$\Theta[V]W = H *_\xi W,$$

where

$$\begin{aligned} H(x,\xi,t) &= \frac{i}{(2\pi)^3}\int_{\mathbb{R}^3_\eta} \frac{V(x+\frac{\hbar}{2m}\eta,t)-V(x-\frac{\hbar}{2m}\eta,t)}{\hbar}\, e^{-i\xi\cdot\eta}\,d\eta \\ &= 16\left(\frac{m}{\hbar}\right)^3 \mathrm{Re}\left\{i\, e^{i\frac{2m}{\hbar}x\cdot\xi}\,\mathcal{F}^{-1}_{x\mapsto\xi}V\left(\frac{2m}{\hbar}\xi,t\right)\right\}. \end{aligned}$$

In fact, it is a simple matter to conclude that

$$|H(x,\xi,t)| \le 16\left(\frac{m}{\hbar}\right)^3 \left|\mathcal{F}^{-1}_{x\mapsto\xi}V\left(\frac{2m}{\hbar}\xi,t\right)\right|, \tag{19}$$

where we denoted $\mathcal{F}^{-1}_{x\mapsto y}f = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3_x} f(x)e^{-ix\cdot y}\,dx$ the inverse Fourier transform of f. We also introduce the following notation for convenience: given $f\in L^1$, define the uniparametric semigroup $G(t)$ acting on f as the integral operator

$$G(t)[f] = \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_z} G(x,\xi,z,v,t)f(z,v)\,dz\,dv\,.$$

Now, the mild WPFP equation (18) may be rewritten as

$$W(t) = G(t)[W_0] - \int_0^t G(t-s)[(H *_\xi W)(s)]\,ds\,. \tag{20}$$

We first proceed to derive *a priori* bounds on W. In the sequel we shall denote by C various positive constants. We have the following

Lemma 3.1 *Let $G(t)$ denote the Green function operator. Let also $1\le p,q<\infty$ and $1+\frac{1}{p}=\frac{1}{r}+\frac{1}{l}, 1+\frac{1}{q}=\frac{1}{s}+\frac{1}{m}$, with $m\le l$. Then, the following estimates* *(i)* $\|G(t)[f]\|_{L^{q,p}} \le e^{6\lambda(1-\frac{1}{m})t}\|G_0\|_{L^{s,r}}\|f\|_{L^{m,l}}$,

(ii) $\|\nabla(G(t)[f])\|_{L^p(\mathbb{R}^3_x;L^q(\mathbb{R}^3_\xi))} \le e^{6\lambda(1-\frac{1}{m})t}\|\nabla G_0\|_{L^r(\mathbb{R}^3_x;L^s(\mathbb{R}^3_\xi))}\|f\|_{L^{m,l}}$,

(iii) $\|H\|_{L^1(\mathbb{R}^3_\xi)} \le C(\|n\|_{L^1(\mathbb{R}^3)} + \|n\|_{L^2(\mathbb{R}^3)}) \le C(\|W\|_{L^1} + \|W\|_{L^{1,2}})$,

hold true. Here, the symbol ∇ denotes any first order derivative. Furthermore, in the particular case $p = q = 1$ in (i) we have

$$\|G(t)[f]\|_{L^1} \le \|f\|_{L^1}\,.$$

Proof. (i) follows from the change of variables $z + \left(\frac{1-e^{-2\lambda t}}{2\lambda}\right)v \mapsto z$, then $e^{-2\lambda t}v \mapsto v$. The proof concludes after application of Young's inequality for the resulting convolution and Minkowski's inequality for the norm of f. The particular case $p = q = 1$ is a direct consequence of Lemma 2.2 (i). The calculations leading to (ii) are analogous to those of (i). Finally, (iii) follows from (19) by the identity (see [1])

$$\begin{aligned}\|\mathcal{F}^{-1}_{x\mapsto y}V(\cdot,t)\|_{L^1(\mathbb{R}^3)} &= \frac{1}{4\pi}\left\|\mathcal{F}^{-1}_{x\mapsto y}\left(\frac{1}{|x|} * n\right)(\cdot,t)\right\|_{L^1(\mathbb{R}^3)} \\ &= \left\|\frac{1}{|\cdot|^2}(\mathcal{F}^{-1}_{x\mapsto y}n)(\cdot,t)\right\|_{L^1(\mathbb{R}^3)}.\end{aligned}$$

Indeed, we first estimate the L^1 norm of $|\cdot|^{-2}(\mathcal{F}^{-1}_{x\mapsto y}n)(\cdot,t)$ outside and inside the 3D unit ball B. We have

$$\left\|\frac{1}{|\cdot|^2}(\mathcal{F}^{-1}_{x\mapsto y}n)(\cdot,t)\right\|_{L^1(\mathbb{R}^3\setminus B)} \le C\|\mathcal{F}^{-1}_{x\mapsto y}n(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le C\|n(\cdot,t)\|_{L^2(\mathbb{R}^3)}\,.$$

Likewise, inside B we get

$$\left\|\frac{1}{|\cdot|^2}(\mathcal{F}^{-1}_{x\mapsto y}n)(\cdot,t)\right\|_{L^1(B)} \le C\|\mathcal{F}^{-1}_{x\mapsto y}n(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \le C\,\|n(\cdot,t)\|_{L^1(\mathbb{R}^3)}.$$

Finally, bounding the norms of n by those of W

$$\|n(\cdot,t)\|_{L^1(\mathbb{R}^3)} \le \|W(t)\|_{L^1}\,, \quad \|n(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le \|W(t)\|_{L^{1,2}}\,,$$

we conclude (iii). ∎

Let $T > 0$. In the sequel we shall manipulate functions $W(x, \xi, t)$ defined on the Banach space $C([0, T]; L^1 \cap L^{1,2})$, endowed with the norm

$$\|W\|_T := \sup_{0 \le t \le T} \Big(\|W(t)\|_{L^1} + \|W(t)\|_{L^{1,2}} \Big) .$$

More precisely, we shall restrict ourselves to the following closed, bounded subset of $C([0, T]; L^1 \cap L^{1,2})$:

$$X_K^T = \Big\{ W \in C([0, T]; L^1 \cap L^{1,2}) : \; W(t = 0) = W_0; \; \|W\|_T \le K \Big\}$$

and define the map $\Gamma : X_K^T \to C([0, T]; L^1 \cap L^{1,2})$ by

$$\Gamma(W)(t) = G(t)[W_0] - \int_0^t G(t - s)[(H *_\xi W)(s)] \, ds .$$

We first notice that Γ is well–defined. Indeed, from Lemma 3.1 (i) and (iii) we have

$$\begin{aligned} &\|G(t)[W_0]\|_{L^1} \le \|W_0\|_{L^1}, \\ &\|G(t - s)[(H *_\xi W)(s)]\|_{L^1} \le C\|W\|_T \, \|W(s)\|_{L^1}, \end{aligned} \tag{21}$$

where we estimated

$$\|H *_\xi W\|_{L^1} \le \|H\|_{L^1(\mathbb{R}^3_\xi)} \|W\|_{L^1}$$

by Young's inequality. On the other hand, the same type of estimates are also true for the $L^{1,2}$ norm. Again from Lemma 3.1 (i) and (iii) we have

$$\begin{aligned} &\|G(t)[W_0]\|_{L^{1,2}} \le \|W_0\|_{L^{1,2}}, \\ &\|G(t - s)[(H *_\xi W)(s)]\|_{L^{1,2}} \le C\|W\|_T \, \|W(s)\|_{L^{1,2}}. \end{aligned} \tag{22}$$

Now, having chosen $K = 2\|W_0\|_T$ and $T \le \dfrac{1}{4CK}$, it is clear that Γ maps X_K^T onto itself and

$$\|\Gamma(W_1) - \Gamma(W_2)\|_T \le \frac{1}{2} \|W_1 - W_2\|_T \quad \forall \, W_1, W_2 \in X_K^T .$$

Hence $\Gamma : X_K^T \to X_K^T$ is a contractive map, so it has a unique fixed point $W \in X_K^T$. This is equivalent to saying that there exists a unique solution $W(t) \in L^1 \cap L^{1,2}$ of the WPFP system (3)–(4) defined on $[0,T]$, for sufficiently small $T > 0$ only depending on W_0. Actually (see [16]), there is a maximum time of existence T_{max} which is either $T_{max} = \infty$ or $T_{max} < \infty$ and $\|W\|_T \to \infty$ when $T \to T_{max}$. In the last section we shall prove that the second possibility cannot occur, hence global existence is attained.

4 Smoothing effects and regularity of the Wigner function

The purpose of this section is to take advantage of the regularization properties of the Fokker–Planck operator in order to derive some smoothing effects on the (Wigner function) solution under the only assumption that the initial data is in $L^1 \cap L^{1,2}$.

We first introduce some useful notations and results.

Definition 4.1 *Let $T > 0$ and f, g be continuous functions in $(0,T)$. We will say that $f(t)$ is equivalent to $g(t)$ at $t = 0$ (and denote it by $f(t) \overset{t=0}{\approx} g(t)$) if there exist three positive constants c_1, c_2 and t_0 such that*

$$c_1\, f(t) \le g(t) \le c_2\, f(t)\,, \qquad \textit{for all } 0 < t \le t_0\,.$$

This concept allows to easily identify the rates of time growth/decay near $t = 0$ of the coefficients of the fundamental solution G_0. We have the following

Lemma 4.1 *Let $a(t)$, $b(t)$, $c(t)$, $d(t)$ and $D(t)$ be given by formulae (16)–(17). Then*

$$D(t) \overset{t=0}{\approx} t^2\,, \quad a(t) \overset{t=0}{\approx} c(t) \overset{t=0}{\approx} \frac{1}{t}\,, \quad d(t) \overset{t=0}{\approx} \frac{1}{t^3}\,, \quad b(t) \overset{t=0}{\approx} 1\,.$$

Proof. For $a(t)$ we observe that $(1/a)(0) = 0$ and $(1/a)'(0) = 4D_{qq}$. Then, a simple integration allows to deduce that

$$\frac{1}{6D_{qq}\,t} \leq a(t) \leq \frac{1}{2D_{qq}\,t}$$

is satisfied in $(0, t_0)$, for some $t_0 > 0$. Analogously, we observe that the first nonvanishing derivative of $1/c$ at $t = 0$ is $(1/c)'(0)$ and that of $1/d$ is $(1/d)'''(0)$. For $b(t)$ we directly check that $0 < b(0) < \infty$. Finally, $D''(0)$ is the first nonvanishing derivative of $D(t)$ at $t = 0$. The proof concludes after integration.

■

We summarize the main regularity properties of $W(x, \xi, t)$, $n(x, t)$ and $V(x, t)$ in the following

Proposition 4.1 *Let $0 < T < T_{max}$ and let also $W(x, \xi, t)$ be the solution of (3)–(4) given by (18). Then,*

(i) $W \in C((0, T); L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))$,

(ii) $W \in C((0, T); W^{1,1} \cap W^{1,\infty}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))$.

Also, the following Hölder regularity is achieved for the density and the potential:

(iii) $n(t) \in C^{0,\alpha}(\mathbb{R}^3_x)$, *for all* $t \in (0, T)$ *and* $0 < \alpha < \frac{1}{2}$,

(iv) $V(t) \in C^{1,\beta}(\mathbb{R}^3_x)$, *for all* $t \in (0, T)$ *and* $0 < \beta < \frac{1}{3}$.

Besides,

(v) $\nabla_x V \in L^2 \cap L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$,

(vi) $\nabla_x n \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$,

(vii) $\xi W \in L^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)^3$ *for all* $t \in (0,T)$. *Actually,*

$$\|\xi W\|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)^3} \leq K\, e^{Ct}\,, \quad 0 < t < T\,, \tag{23}$$

where K *and* C *are positive constants depending on* $\|W\|_T$.

Proof. The first result is reached as a consequence of Lemmata 2.2 and 3.1 by using that $d(t) \stackrel{t=0}{\sim} t^{-3}$, then $d(t)^{\frac{1}{p'}} \in L^1(0,T)$ for $1 \leq p < \frac{3}{2}$. We prove (i) in four steps. The *first step* consists of estimating

$$\begin{aligned}\|W(t)\|_{L^p} &\leq \|G(t)[W_0]\|_{L^p} + \int_0^t \|G(t-s)[(H*W)(s)]\|_{L^p}\, ds \\ &\leq C\|W_0\|_{L^1}\, d(t)^{\frac{1}{p'}} + C\|W\|_T^2 \int_0^t d(s)^{\frac{1}{p'}}\, ds.\end{aligned}$$

Then, we deduce that $W \in C((0,T_{max}); L^p)$ for all $1 \leq p < \frac{3}{2}$. In the *second step* we start from an arbitrarily small time $\epsilon > 0$, that is, rewrite $W(t)$ for $t > \epsilon$ as

$$W(t) = G(t-\epsilon)[W(\epsilon)] - \int_\epsilon^t G(t-s)[(H*W)(s)]\, ds,$$

then we estimate $\|W(t)\|_{L^q}$ with $q < 3$ as

$$\|W(t)\|_{L^q} \leq Cd(t-\epsilon)^{\frac{1}{p'}}\|W(\epsilon)\|_{L^p} + C\|W\|_T \int_\epsilon^t d(t-s)^{\frac{1}{p'}}\|W(s)\|_{L^p}\, ds.$$

The *third step* is analogous. Indeed, for $r < \infty$ and $t > 2\epsilon$ we can now estimate $\|W(t)\|_{L^r}$ as

$$\|W(t)\|_{L^r} \leq Cd(t-2\epsilon)^{\frac{1}{p'}}\|W(2\epsilon)\|_{L^q} + C\|W\|_T \int_{2\epsilon}^t d(t-s)^{\frac{1}{p'}}\|W(s)\|_{L^q}\, ds\,.$$

Finally, in a *fourth step* we obtain a uniform bound for W by writing

$$\|W(t)\|_{L^\infty} \leq Cd(t-3\epsilon)^{\frac{1}{p'}}\|W(3\epsilon)\|_{L^r} + C\|W\|_T \int_{3\epsilon}^t d(t-s)^{\frac{1}{p'}}\|W(s)\|_{L^r}\, ds$$

for any $t > 3\epsilon$. Notice that p, q and r are linked by Young's relations at every step. The arbitrariness of ϵ allows us to conclude. Also note that we have repeatedly used the property $G(t)[G(s)[f]] = G(t+s)[f]$ of evolution semigroups. Finally, Pazy's results (see § 6 of [16]) ensure the continuity of the Green function operator $G(t)$, thus of the Wigner function.

To prove (ii), we first take gradients in the mild equation (20) and obtain

$$\nabla_{(x,\xi)} W(t) = \nabla_{(x,\xi)} G(t)[W_0] - \int_0^t \nabla_{(x,\xi)} G(t-s)[H * W(s)]\, ds\,.$$

As for (i) we can prove (ii) in several steps, depending on the time integrability of $\|\nabla_{(x,\xi)} G(x,\xi,z,v,t)\|_{L^p} = \|\nabla_{(x,\xi)} G_0(x,\xi,t)\|_{L^p}$. Using Lemma 2.2 (iii) with $q = p$ and Lemma 4.1 we conclude that

$$\|\nabla_{(x,\xi)} G_0(x,\xi,t)\|_{L^p} \le C\, t^{\frac{3}{p}-\frac{7}{2}}.$$

Therefore, $\|\nabla_{(x,\xi)} G_0(x,\xi,t)\|_{L^p} \in L^1(0,T)$ for $1 \le p < \frac{6}{5}$. Using now the same ideas than before, (ii) can be reached in seven steps. The time continuity is deduced as before.

To prove (iii) we first observe that Morrey's Theorem (see for example [3]) yields

$$|n(x,t) - n(y,t)| \le \|\nabla_x n(\cdot,t)\|_{L^p(\mathbb{R}^3)} |x-y|^{1-\frac{3}{p}}, \qquad \text{for } p > 3\,.$$

Then, it suffices to control $\|\nabla_x n(\cdot,t)\|_{L^p(\mathbb{R}^3)}$ for some $p > 3$. To this aim, we shall show that $\|\nabla_x W(t)\|_{L^p(\mathbb{R}^3_x;L^1(\mathbb{R}^3_\xi))}$ is bounded. By using Lemmata 2.2 (iii) and 3.1, we have

$$\begin{aligned}
\|\nabla_x W(t)\|_{L^p(\mathbb{R}^3_x;L^1(\mathbb{R}^3_\xi))} &\le \|\nabla_x G_0(t)\|_{L^q(\mathbb{R}^3_x;L^1(\mathbb{R}^3_\xi))} \|W_0\|_{L^{1,2}} \\
&\quad + \|W\|_T^2 \int_0^t \|\nabla_x G_0(s)\|_{L^q(\mathbb{R}^3_x;L^1(\mathbb{R}^3_\xi))}\, ds \\
&\le \|W_0\|_{L^{1,2}}\, t^{\frac{3}{2q}-2} + C\|W\|_T^2\, t^{\frac{3}{2q}-1}
\end{aligned}$$

with $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{2}$. Now, choosing $\frac{6}{5} < q < \frac{3}{2}$ and then $3 < p < 6$ we get the desired bound with $\alpha = 1 - \frac{3}{p}$. We notice that the above inequality is still valid for $p = 2$ $(q = 1)$, then assertion (vi) holds.

We prove (iv) by using the convolution form (5) of V and splitting the integral into two parts:

$$\begin{aligned} |V(x,t)-V(z,t)| &\leq C\int_{\mathbb{R}^3}\frac{|n(x-y,t)-n(z-y,t)|}{|y|}\,dy \leq \int_{|y|<R}+\int_{|y|\geq R} \\ &\leq C\|\nabla_x n(\cdot,t)\|_{L^p}|x-z|^{1-\frac{3}{p}}\int_{|y|<R}\frac{1}{|y|}\,dy \\ &\quad +\frac{1}{R}\int_{\mathbb{R}^3}|n(x-y,t)+n(z-y,t)|\,dy \\ &\leq CR^2\|\nabla_x n(\cdot,t)\|_{L^p}|x-z|^{1-\frac{3}{p}}+\frac{2Q}{R}\,. \end{aligned}$$

Then, optimizing over R we get

$$|V(x,t)-V(z,t)|\leq CQ^{\frac{2}{3}}\|\nabla_x n(\cdot,t)\|_{L^p}^{\frac{1}{3}}|x-z|^{\frac{1}{3}-\frac{1}{p}}\,,$$

thus the continuity of V. For the first order derivative we may analogously write

$$\begin{aligned} |\nabla_x V(x,t)-\nabla_x V(z,t)| &\leq C\int_{\mathbb{R}^3}\frac{|n(x-y,t)-n(z-y,t)|}{|y|^2}\,dy \\ &\leq CQ^{\frac{1}{3}}\|\nabla_x n(\cdot,t)\|_{L^p}^{\frac{2}{3}}|x-z|^{\frac{2}{3}-\frac{2}{p}}\,, \end{aligned}$$

which yields the Hölder continuity of $\nabla_x V$. This concludes the proof of (iv) with $\beta=\frac{2}{3}-\frac{2}{p}$.

To prove (v), we first notice that

$$\|\nabla_x V\|_{L^2}\leq C\Big(\|n\|_{L^1}+\|n\|_{L^2}\Big).$$

Also, the boundedness of $\nabla_x V$ is a straightforward consequence of $\nabla_x V\in L^2$ and the Hölder regularity $\nabla_x V\in C^{0,\beta}$.

Finally, (vii) follows by multiplying Eq. (18) against ξ and taking L^1 norms. Then, using (vi), Lemma 4.1 and the fact that $\|\xi H\|_{L^1}\leq C\Big(\|W\|_T+\|\nabla_x n\|_{L^2}\Big)$ we get

$$\|\xi W(\cdot,\cdot,t)\|_{L^1}\leq \sup_{0<t<T}F(t)+C\|W\|_T\int_0^t e^{-2\lambda(t-s)}\|\xi W(\cdot,\cdot,s)\|_{L^1}\,ds\,,$$

with

$$F(t) = C\|W_0\|_{L^1}\sqrt{t} + e^{-2\lambda t}\|\xi W_0\|_{L^1} + C\|W\|_T^2\, t^{\frac{3}{2}} + C\|W\|_T^2\, t + \int_0^t e^{-2\lambda(t-s)}\,\|\nabla_x n(\cdot,s)\|_{L^2}\,ds\,.$$

Now, Gronwall's inequality applies to yield (23).

■

5 Existence of global solutions

This section is devoted to prove that the solution obtained in Section 3 is actually defined in $[0,\infty)$, that is, $T_{max} = \infty$. To this aim, we shall equivalently show that the norm in $L^1 \cap L^{1,2}$ cannot blow up in finite time.

We start with some considerations concerning the kinetic energy of the system.

Lemma 5.1 *Consider the electron kinetic energy associated with* $f(x,\xi,t)$ *to be defined by*

$$E[f](t) = \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_\xi} \frac{|\xi|^2}{2} f(x,\xi,t)\,d\xi\,dx\,.$$

Let W *and* W^H *be the solution of the WPFP system (3)–(7) and its corresponding Hussimi transform (cf. (9)), respectively. The following assertions hold true:*

(i) $E[W](t) < +\infty$ *for all* $0 < t < T$.

(ii) $E[W]$ *solves*

$$\frac{d}{dt}\Big(E[W](t) + \frac{1}{2m}\|\nabla_x V(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\Big) = 3\frac{D_{pp}}{m^2}Q - 4\lambda E[W](t) - \frac{D_{qq}}{m}\|n(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\,.$$

(iii) The Husimi kinetic energy $E[W^H]$ *is connected to* $E[W]$ *through the following relation*

$$E[W^H](t) = E[W](t) + \frac{3\hbar}{2m} Q \,.$$

(iv) $E[W]$ *is bounded from below. In fact,*

$$E[W](t) \geq -\frac{3\hbar}{2m} Q \,, \quad \forall t > 0 \,.$$

Proof. (i) follows from Eq. (18) by integrating against $|\xi|^2$ and estimating

$$\begin{aligned}
E[W](t) \;&\leq\; e^{-4\lambda t} E[W](0) + \| |\xi|^2 G_0(t) \|_{L^1} \|n\|_{L^1} \\
&\quad + \int_0^t \| |\xi|^2 G_0(t-s) \|_{L^1} \| (H *_\xi W)(s) \|_{L^1} \, ds \\
&\quad + \int_0^t e^{-4\lambda(t-s)} \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_\xi} |\xi|^2 (H *_\xi W) \, d\xi \, dx \, ds \\
&\leq\; E[W](0) + C\, t \, \|n\|_{L^1} + C\, t^2 \|W\|^2_T \\
&\quad + \int_0^t \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_\xi} |\xi|^2 (H *_\xi W) \, d\xi \, dx \, ds \\
&\leq\; C(T) + \int_0^t \Big(\int_{\mathbb{R}^3_x} J \cdot \nabla_x V \, dx \Big) \, ds,
\end{aligned}$$

where we have used again Lemma 4.1. Then, Proposition 4.1 (v) and (vii) allows to conclude.

Multiplying Eq. (3) by $|\xi|^2$, integrating against x and ξ and using the Poisson equation $\Delta_x V = n$ leads to *(ii)*. *(iii)* follows from a straightforward calculation. *(iv)* is a simple consequence of *(iii)* given that the Husimi function W^H is positive. ∎

We first remark that the density function $n(x,t)$ is nonnegative (see for example [1, 14]). Then, integrating Eq. (18) in x and ξ shows that the total charge of the system

$$Q = \int_{\mathbb{R}^3_x} n(x,t) \, dx = \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_\xi} W(x,\xi,t) \, d\xi \, dx$$

is preserved along the time evolution.

Now we are ready to finish the proof of Theorem 1.1. We only need to show the existence of a solution as stated in the theorem. To this aim, estimating as in (21) and (22) and using Lemma 3.1 (iii), we find

$$\|W(t)\|_{L^1} + \|W(t)\|_{L^{1,2}} \leq \|W_0\|_{L^1} + \|W_0\|_{L^{1,2}}$$
$$+C\int_0^t \Big(\|n(s)\|_{L^1(\mathbb{R}^3)} + \|n(s)\|_{L^2(\mathbb{R}^3)}\Big)\Big(\|W(s)\|_{L^1} + \|W(s)\|_{L^{1,2}}\Big)\, ds\,.$$

Now it is enough to prove that

$$\int_0^t \Big(\|n(s)\|_{L^1(\mathbb{R}^3)} + \|n(s)\|_{L^2(\mathbb{R}^3)}\Big)\, ds$$

is finite for finite time, as in that case we finish by using Gronwall's lemma. As $\|n(t)\|_{L^1(\mathbb{R}^3)} = Q$ is constant in time, the problem is reduced to showing that $\int_0^t \|n(s)\|_{L^2(\mathbb{R}^3)}\, ds$ remains bounded on bounded time intervals.

Now, integrating (i) in Lemma 5.1 between 0 and t and using the lower bound for the kinetic energy given in Lemma 5.1 (iii), we get

$$\int_0^t \|n(s)\|_{L^2(\mathbb{R}^3)}\, ds \leq t + \int_0^t \|n(s)\|^2_{L^2(\mathbb{R}^3)}\, ds \leq C(W_0) + C(\lambda, D_{pp}, D_{qq})\; t$$

after some simple estimates. This implies that $\int_0^t \|n(s)\|_{L^2(\mathbb{R}^3)}\, ds$ cannot blow up at finite time. Now we are done with the proof of Theorem 1.1.

Remark 1 *Notice that the same proof applies to the general WPFP equation with nonvanishing friction* (1)*, with inessential modifications (in the sense of estimates and time integrability) in the expression for the fundamental solution due to the additional term* $\frac{2D_{pq}}{m}\,\mathrm{div}_x(\nabla_\xi W)$*. In fact, we find again*

$$G(x,\xi,z,v,t) = G_0\Big(x - z - \Big(\frac{1-e^{-2\lambda t}}{2\lambda}\Big)v, \xi - e^{-2\lambda t}v, t\Big),$$

where now

$$G_0(x,\xi,t) = \delta(t)\exp\Big\{-\alpha(t)|x|^2 + \beta(t)(x\cdot\xi) - \gamma(t)|\xi|^2\Big\}$$

with

$$\alpha(t) = m^2\lambda^3 D_{pp}\ \frac{(1-e^{-4\lambda t})}{\Delta(t)}\,,$$

$$\beta(t) = m^2\lambda^2\ \frac{\Big(D_{pp}(1-e^{-2\lambda t})^2 + 8m\lambda^2 D_{pq} t\Big)}{\Delta(t)}\,,$$

$$\gamma(t) = \frac{m^2\lambda D_{pp}}{4}\ \frac{\Big(4\lambda t\Big(1+4\lambda^2 m^2\frac{D_{qq}}{D_{pp}}\Big) - (1-e^{-2\lambda t})(3-e^{-2\lambda t})\Big)}{\Delta(t)}\,,$$

$$\delta(t) = \left(\frac{m^2\lambda^2}{\pi D_{pp}\sqrt{\Delta(t)}}\right)^3,$$

$$\Delta(t) = D_{pp}^2 D(t) - 4m\lambda^2 D_{pq} t\Big(D_{pp}(1-e^{-2\lambda t})^2 + m\lambda^2 D_{pq}\Big)\,.$$

On the contrary, the ideas employed in the proof of Theorem 1.1 cannot be extended to the frictionless WPFP system (2) because of the lack of elliptic regularization in the $x-$direction. This problem will be tackled by the authors in a forthcoming paper.

Remark 2 *The regularity properties proved in Theorem 1.1 allow to rigorously justify all the a priori estimates derived on the Wigner function, the density and the potential. In particular, the energy equation established in Lemma 5.1 (i) makes full sense. Indeed, it is clear that*

$$\|n(t)\|_{L^2(\mathbb{R}^3)} \le \|W(t)\|_{L^{1,2}}\,.$$

Also, from standard elliptic estimates we have

$$\|\nabla_x V(t)\|_{L^2(\mathbb{R}^3)} \le C\,\|n(t)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \le C\,Q^{\frac{2}{3}}\|n(t)\|^{\frac{1}{3}}_{L^2(\mathbb{R}^3)}\,.$$

Remark 3 *Some (exponential) control of the growth in time of the kinetic energy is also possible. Indeed, one can easily deduce from the energy equation stated in Lemma 5.1 (i) the following bound*

$$E[W](t) \le C(W_0) + 3\frac{D_{pp}}{m^2}Q\,t + 4\lambda\int_0^t |E[W](s)|\,ds\,.$$

Then, once we know from Lemma 5.1 (iii) that $E[W]$ cannot be "very negative", it is clear that a (sufficiently large) positive constant (denoted again by $C(W_0)$ for simplicity) must exist such that the above inequality is still valid for $|E[W](t)|$. Consequently, Gronwall's lemma applies to give

$$E[W](t) \leq C(W_0, \lambda, D_{pp})\Big(1 + t\, e^{4\lambda t}\Big).$$

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