

MSM3P22/MSM4P22  
Further Complex Variable Theory & General Topology  
Course notes - Handout 7

José A. Cañizo

November 2, 2012

A topology is a certain kind of additional *structure* on a set. As it happens with groups or vector spaces, there are a number of constructions one can carry out with topologies: we already studied *subspaces* when we looked at the induced topology on a subset of a topological space. *Products* and *quotients* can also be defined, and it is what we will do in the next classes.

## 7.1 The product topology

**Definition 7.1** (Product of two spaces). Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be two topological spaces. Their *product* topological space is the set  $X \times Y$  with the topology whose base is

$$\mathcal{B} = \{U \times V \subseteq X \times Y \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}. \quad (1)$$

**Lemma 7.2.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be two topological spaces. The collection

$$\mathcal{S} = \{U \times Y \mid U \in \mathcal{T}_X\} \cup \{X \times V \mid V \in \mathcal{T}_Y\}$$

is a subbase for the product topology of  $X \times Y$ .

**Lemma 7.3.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be two topological spaces. If  $\mathcal{B}_X$  is a base for  $\mathcal{T}_X$  and  $\mathcal{B}_Y$  is a base for  $\mathcal{T}_Y$ , then

$$\mathcal{B} := \{B_1 \times B_2 \mid B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\}$$

is a base for the product topology of  $X \times Y$ .

**Exercise 7.4.** 1. Show that the collection  $\mathcal{B}$  from equation (1) is indeed a base for some topology on  $X \times Y$ . Is  $\mathcal{B}$  a topology on  $X \times Y$ ?

2. Prove Lemma 7.3.

It is easy to define the product of a finite number of spaces by iterating the above definition. In order to do that it is important to know that *the order* in which one takes the product does not matter:

**Lemma 7.5.** *Let  $X, Y, Z$  be topological spaces. The usual identification between sets  $(X \times Y) \times Z$  and  $X \times (Y \times Z)$  is a homeomorphism.*

Let us generalise this to products of an arbitrary number of sets:

**Definition 7.6.** Consider a collection  $\{(X_\alpha, \mathcal{T}_\alpha) \mid \alpha \in I\}$  of topological spaces indexed in a set  $I$ . Their *product topological space* is the set  $\prod_{\alpha \in I} X_\alpha$  with the topology whose base is

$$\mathcal{B} = \left\{ \prod_{\alpha \in I} U_\alpha \mid U_\alpha \in \mathcal{T}_\alpha \text{ for all } \alpha \in I, \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha. \right\}$$

**Definition 7.7** (Coordinates and projections). If  $x$  is a point in the cartesian product  $\prod_{\alpha \in I} X_\alpha$ , then by definition  $x$  is a list of the form  $(x_\alpha)_{\alpha \in I}$ . For any given  $\beta \in I$ , We call  $x_\beta$  its  $\beta$  *coordinate*, and for each  $\beta \in I$  the function

$$\pi_\beta : \prod_{\alpha \in I} X_\alpha \rightarrow X_\beta, \quad \text{given by } (x_\alpha)_{\alpha \in I} \mapsto x_\beta,$$

which associates to each  $(x_\alpha)_{\alpha \in I}$  its  $\beta$  coordinate, is called the  $\beta$  projection.

### 7.1.1 Some properties of products

**Theorem 7.8.** *Consider a collection  $\{(X_\alpha, \mathcal{T}_\alpha) \mid \alpha \in I\}$  of topological spaces.*

1. *Let  $A_\alpha$  be a subset of  $X_\alpha$  for each  $\alpha \in I$ . Then*

$$\overline{\prod_{\alpha \in I} A_\alpha} = \prod_{\alpha \in I} \overline{A_\alpha}.$$

2. *Let  $A_\alpha$  be a subset of  $X_\alpha$  for each  $\alpha \in I$ . On  $A_\alpha$  we consider the induced topology from  $X_\alpha$ . Then the following two topologies on the set  $\prod_{\alpha \in I} A_\alpha$  are the same:*

(a) *The product topology, taking on each  $A_\alpha$  the induced topology from  $X_\alpha$ .*

(b) *The topology on  $\prod_{\alpha \in I} A_\alpha$  induced from  $\prod_{\alpha \in I} X_\alpha$ .*

3. *If  $X_\alpha$  is Hausdorff for all  $\alpha \in I$ , then  $\prod_{\alpha \in I} X_\alpha$  is Hausdorff.*

4. *Let  $Y$  be a topological space. A function  $f : Y \rightarrow \prod_{\alpha \in I} X_\alpha$  is continuous if and only if its coordinates are continuous.*

5. *A sequence in  $\prod_{\alpha \in I} X_\alpha$  converges if and only if each of its coordinates converges.*

**Exercise 7.9.** *Prove points 1, 3 and 5 of the previous Theorem.*