MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Course notes - Handout 9

José A. Cañizo

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9.1 Compactness

We will study now a useful generalization of some of the properties of a closed bounded interval of $\mathbb{R}$:

Definition 9.1 (Covering, open covering)**.** Let X be a topological space. A *covering* of X is a collection $\mathscr{A}$ of subsets of X such that

$$\bigcup_{A \in \mathscr{A}} A = X.$$

A covering $\mathscr{A}$ of X is said to be *open* if each of the elements of $\mathscr{A}$ is an open set.

Definition 9.2 (Compact space)**.** A topological space X is said to be *compact* if every open covering $\mathscr{A}$ of X contains a finite subcollection that is also a covering of X.

Accordingly, if $A \subseteq X$, we say that A is a compact subset of X when A is compact with the induced topology (this is just the previous definition applied to A.) Notice that this can be said in other words:

Definition 9.3 (Covering, open covering of a subset)**.** Let X be a topological space and $K \subseteq X$. A *covering* of K by subsets of X is a collection $\mathscr{A}$ of subsets of X such that

$$\bigcup_{A \in \mathscr{A}} A \supseteq K.$$

A covering $\mathscr{A}$ of K by open subsets of X is a covering of K by subsets of X which are open.

Lemma 9.4 (Compact subset of a space)**.** *A subset K of a topological space X is compact if and only if every covering of K by open subsets of X contains a finite subcollection that is also a covering of K.*

Exercise 9.5. *Prove Lemma 9.4.*

Because of this equivalence, coverings of a subset K by open subsets of X (Def. 9.3) and coverings of K by open subsets of K (Def. 9.1) are often used interchangeably. We will work with the one which is most convenient for the particular setting we discuss at each point.

Lemma 9.6. *A closed subset of a compact topological space is compact.*

Sketch of proof. Denote the subset by C and the space by X. If you have an open cover of C you can always get an open cover of X by adding $X \setminus C$ to it; extracting a finite subcover of this you get a finite subcover for C. □

Lemma 9.7. *A compact subset of a Hausdorff topological space is closed.*

Sketch of proof. Take any point y not in the subset, which we call C. Since the space is Hausdorff, any point x in C can be "separated" from y by disjoint open subsets U_x, V_x such that $x \in U_x$, $y \in V_x$. Then the set of all U_x is a cover of C, from which you can keep only a finite number that still cover C. Then the intersection of the V_x corresponding to this finite family is an open set which contains x and does not intersect C. □

Theorem 9.8. *The image of a compact space under a continuous map is compact.*

Sketch of proof. Let $f : X \to Y$ be continuous. If you have an open cover of $f(Y)$, then its inverse image by f is an open cover of X. Extract a finite subfamily of that one, and the corresponding sets in Y should cover $f(Y)$. □

Theorem 9.9. *The product of a finite number of compact spaces is compact.*

9.2 Compactness of subsets of $\mathbb{R}^d$

Theorem 9.10. *The set* $[0, 1]$ *with the usual topology is compact.*

Proof. Take any open cover $\mathscr{A}$ of $[0, 1]$ by open subsets of $\mathbb{R}$. We consider the number x^* defined by

$$\begin{aligned} &B := \{x \in (0, 1] \mid [0, x] \text{ can be covered by a finite number of sets in } \mathscr{A}.\} \\ &x^* := \sup B. \end{aligned}$$

First, note that x^* is well defined, since:

1. B is bounded above by 1.
2. B is not empty: the point $0 \in [0, 1]$ has to be in some open set U of $\mathscr{A}$, and the interval $[0, \epsilon]$ must be contained in U for ϵ small enough. Hence $[0, \epsilon]$ can be covered by just one set in $\mathscr{A}$.

Since $x^* \in [0, 1]$ there must exist $U \in \mathscr{A}$ such that $x^* \in U$. Then, there must be some $\epsilon > 0$ for which $(x^* - \epsilon, x^* + \epsilon) \subseteq U$. Two things can happen:

1. If $x^* < 1$, take $x \in B$ such that $x \in (x^* - \epsilon, x^*]$. Then the interval $[0, x^* + \epsilon)$ can be covered by a finite number of sets in $\mathscr{A}$: those that covered $[0, x]$, plus U. This contradicts the definition of x^*.
2. If $x^* = 1$, then again take $x \in B$ such that $x \in (x^* - \epsilon, x^*]$. Then the interval $[0, 1]$ can be covered by a finite number of sets in $\mathscr{A}$: those that cover $[0, x]$, plus U, which finishes the proof.

□

Definition 9.11. A subset $A \subseteq \mathbb{R}^d$ is *bounded* if there exists $R > 0$ such that $A \subseteq B(0, R)$ (considering the usual Euclidean distance.)

Theorem 9.12. *A subset K of $\mathbb{R}^d$ with the usual topology is compact if and only if it is closed and bounded.*

Sketch of proof. $\overline{B(0, R)}$ is compact because it is contained in $[-R, R]^d$ (due to Theorems 9.10 and 9.9, and Lemma 9.6). A closed and bounded subset of $\mathbb{R}^d$ is a closed subset of $\overline{B(0, R)}$ for some R, which is compact by Lemma 9.6.

A compact subset of $\mathbb{R}^d$ must be closed due to Lemma 9.7. It is easy to see that it must also be bounded. □

Corollary 9.13. *Let $f : K \to \mathbb{R}$ be a continuous function from a compact topological space K to $\mathbb{R}$. Then the image of f is bounded and f reaches its maximum at some point in K: there is $x \in K$ such that $f(x) \geq f(y)$ for all $y \in K$.*

Sketch of proof. Theorem 9.8 implies that the image of f must be compact, and then Theorem 9.12 shows that $f(K)$ is bounded. Being compact, $f(K)$ must have a largest element, for otherwise $\{(-\infty, x) \mid x \in f(K)\}$ is an open covering of $f(K)$ from which one cannot extract a finite subcovering, reaching a contradiction. □

9.3 Sequential compactness

Definition 9.14. We say that a topological space X is *sequentially compact* when every sequence in X has a subsequence which converges to a point in X.

Theorem 9.15. *Let X be a metrizable space. Then it is compact if and only if it is sequentially compact.*

Proof that compactness implies sequential compactness. (We omit the proof that sequential compactness implies compactness.) Take d a distance for the topology of X and consider a sequence $\{x_n\}_{n \geq 1}$ in X. Reasoning by contradiction, assume that no subsequence of $\{x_n\}$ converges to a point in X. Then for every point $x \in X$ there must be $\epsilon_x > 0$ such that $B(x, \epsilon_x)$ contains only a finite number of terms of the sequence (otherwise one can build a subsequence that converges to x). Then $\{B(x, \epsilon_x) \mid x \in X\}$ is an open covering of X, of which we can extract a finite subcovering. This implies that the sequence can only have a finite number of terms, which is a contradiction. □