

MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Solutions to Problem sheet 3

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Unless otherwise specified, the symbols X , Y and Z represent topological spaces in the following exercises.

Exercise 3.1. Let $A \subseteq B \subseteq X$ be subsets of X . As usual, on B we consider the topology induced from X . Show that the topology on A induced from B is the same as the topology on A induced from X .

Solution. Denote by \mathcal{T} the topology of X . The topology on A induced from X is defined by

$$\mathcal{T}_A^X := \{U \cap A \mid U \in \mathcal{T}\}.$$

The induced topology on B is defined by

$$\mathcal{T}_B := \{U \cap B \mid U \in \mathcal{T}\},$$

and the topology on A induced from B is defined by

$$\mathcal{T}_A^B := \{V \cap A \mid V \in \mathcal{T}_B\}.$$

Hence we have

$$\begin{aligned} \mathcal{T}_A^B &= \{V \cap A \mid V \in \mathcal{T}_B\} = \{(U \cap B) \cap A \mid U \in \mathcal{T}\} \\ &= \{U \cap (B \cap A) \mid U \in \mathcal{T}\} = \{U \cap A \mid U \in \mathcal{T}\} = \mathcal{T}_A^X. \end{aligned}$$

Note that in the last step we used that $A \subseteq B$ to say that $A \cap B = A$.

Exercise 3.3. Show that the countable collection

$$\{(a, b) \times (c, d) \mid a, b, c, d \in \mathbb{Q} \text{ and } a < b, c < d\}$$

is a basis for the usual topology on \mathbb{R}^2 .

Solution. We recall that the usual topology in \mathbb{R}^2 is defined to be the set of all open sets, where a set being “open” means that it contains a ball surrounding each of its points (see 1.5.1 in Handout 1).

Denote our candidate base by \mathcal{B} . First, note that of course every set in \mathcal{B} is open, as can be easily checked. In order to show that \mathcal{B} is a base for the usual topology, take any open set $U \subseteq \mathbb{R}^2$. Then for each $x = (x_1, x_2) \in U$ there exists an open ball $B(x, r_x)$ of radius $r_x > 0$ such that $B(x, r_x) \subseteq U$. By making r_x slightly smaller if necessary we can also assume that $r_x \in \mathbb{Q}$ (we can just choose any positive rational number smaller than r_x). Since for all $x \in U$ we have

$$\left(x_1 - \frac{r_x}{2}, x_1 + \frac{r_x}{2}\right) \times \left(x_2 - \frac{r_x}{2}, x_2 + \frac{r_x}{2}\right) \subseteq B(x, r_x) \subseteq U,$$

we can write

$$U = \bigcup_{x \in U} \left(x_1 - \frac{r_x}{2}, x_1 + \frac{r_x}{2}\right) \times \left(x_2 - \frac{r_x}{2}, x_2 + \frac{r_x}{2}\right).$$

Hence, every open set U can be written as a union of elements of \mathcal{B} , showing that it is a base.

Exercise 3.8. (1 mark) Show that a function $f : X \rightarrow Y$ is continuous if and only if

$$f^{-1}(\text{int}(A)) \subseteq \text{int}(f^{-1}(A))$$

for all sets $A \subseteq Y$.

Solution. Assume first that f is continuous, and take any set $A \subseteq Y$. Since $\text{int}(A)$ is open we have that $f^{-1}(\text{int}(A))$ is an open set such that

$$f^{-1}(\text{int}(A)) \subseteq f^{-1}(A).$$

By definition of the interior we deduce that

$$f^{-1}(\text{int}(A)) \subseteq \text{int}(f^{-1}(A)).$$

For the other implication, assume that for all subsets $A \subseteq Y$ it holds that

$$f^{-1}(\text{int}(A)) \subseteq \text{int}(f^{-1}(A)).$$

Take any open set $U \subseteq Y$. Taking $A = U$ above, and using that $\text{int}(U) = U$ we have

$$f^{-1}(U) \subseteq \text{int}(f^{-1}(U)).$$

But in general it is true that $\text{int}(f^{-1}(U)) \subseteq f^{-1}(U)$; since both inclusions hold, we deduce that $\text{int}(f^{-1}(U)) = f^{-1}(U)$. Since the interior of something is always open, $f^{-1}(U)$ is open. This shows that f is continuous.

Exercise 3.9. Find a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is continuous at exactly one point.

Solution. An example of this is

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

It is not difficult to check that it is continuous only at $x = 0$.

Exercise 3.11. Let $f : X_1 \rightarrow Y_1$ and $g : X_2 \rightarrow Y_2$ be continuous functions between topological spaces. Define

$$\begin{aligned} f \times g : X_1 \times X_2 &\rightarrow Y_1 \times Y_2 \\ (x_1, x_2) &\mapsto (f(x_1), g(x_2)). \end{aligned}$$

Show that $f \times g$ is continuous.

Solution. (It is assumed here that the topology we should consider on $X_1 \times X_2$ and $Y_1 \times Y_2$ is the product topology.) By Exercise 4.3 in Handout 4, in order to show that $f \times g$ is continuous it is enough to check that $(f \times g)^{-1}(B)$ is open for each B in the base \mathcal{B} given by

$$\mathcal{B} := \{U_1 \times U_2 \mid U_1 \text{ is open in } Y_1, U_2 \text{ is open in } Y_2\}.$$

(\mathcal{B} is a base for the topology of $Y_1 \times Y_2$ by definition of the product topology.) Take then a set $B = U_1 \times U_2 \in \mathcal{B}$. From the definition of $f \times g$ it is easy to see that

$$(f \times g)^{-1}(B) = (f \times g)^{-1}(U_1 \times U_2) = f^{-1}(U_1) \times g^{-1}(U_2).$$

Since both f and g are continuous, we have that $f^{-1}(U_1)$ is open in X_1 and $f^{-1}(U_2)$ is open in X_2 . By definition of the product topology of $X_1 \times X_2$ we see that $f^{-1}(U_1) \times g^{-1}(U_2)$ is open and this finishes the proof.

Exercise 3.12. Let $f : X \times Y \rightarrow Z$ be a continuous function. Show that f is continuous in each variable separately; this is: for each $x_0 \in X$, the function $h : Y \rightarrow Z$ defined by $h(y) = f(x_0, y)$ is continuous; and for each $y_0 \in Y$, the function $g : X \rightarrow Z$ defined by $h(x) = f(x, y_0)$ is continuous.

Is it true in general that if a function $f : X \times Y \rightarrow Z$ is continuous in each variable separately, then it is continuous?

Solution. Take $x_0 \in X$ and consider the function $p_{x_0} : Y \rightarrow X \times Y$ defined by

$$p_{x_0}(y) = (x_0, y) \quad \text{for } y \in Y.$$

This function is clearly continuous due to the previous exercise. The function h in the statement can be written as $h = f \circ p_{x_0}$, a composition of continuous functions and hence continuous. An analogous reasoning applies to g .

The fact that a function is continuous in each variable separately does not imply that it is continuous. A well-known example of this is the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0), \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

A good explanation of this and further examples can be found [at this link](#).

Exercise 3.14. Let X be a metric space with metric d .

1. Show that $d : X \times X \rightarrow \mathbb{R}$ is continuous.
2. Let \mathcal{T} be a topology on X . Assume that $d : (X, \mathcal{T}) \times (X, \mathcal{T}) \rightarrow \mathbb{R}$ is continuous. Show that the topology \mathcal{T} is finer than the metric topology of X .

Solution. 1. Since both $X \times X$ and \mathbb{R} are metric spaces we can use the metric characterization of continuity (a metric on $X \times X$ which generates the product topology is for example $\tilde{d}((x, y), (z, w)) = d(x, z) + d(y, w)$, for any $x, y, z, w \in X$.) Recall that in a metric space we have

$$d(x, y) \geq |d(x, z) - d(z, y)|$$

for any three points $x, y, z \in X$. Using this, for any two points $(x, y), (z, w) \in X \times X$ we have

$$\begin{aligned} |d(x, y) - d(z, w)| &\leq |d(x, y) - d(x, w)| + |d(x, w) - d(z, w)| \\ &\leq d(y, w) + d(x, z) = \tilde{d}((x, y), (z, w)). \end{aligned}$$

This shows that d is a Lipschitz function, and in particular continuous.

2. Take any point $x \in X$. From the previous exercise, the function $d_x : (X, \mathcal{T}) \rightarrow \mathbb{R}$ given by $d_x(y) = d(x, y)$ is continuous, so for any $r > 0$ the set $d_x^{-1}((-r, r)) \subseteq X$ is an open set in \mathcal{T} . But

$$d_x^{-1}((-r, r)) = \{y \in X \mid d(x, y) < r\} = B(x, r),$$

the open ball centered at x with radius r . Hence, all open balls are in \mathcal{T} and in particular the topology \mathcal{T} is finer than the metric topology (for example, by using Lemma 4.7 in Handout 4).

Exercise 3.15. Let A be a subset of X . If d is a metric for the topology of X , show that $d|_{A \times A}$ is a metric for the induced topology on A .

Solution. Call \mathcal{T} the topology of X . Recall that “ d is a metric for the topology of X ” means that the metric topology associated to d is precisely \mathcal{T} . Also, “the metric topology associated to d ” is the topology given by

$$\mathcal{T}_d := \{U \subseteq X \mid \forall x \in U \exists r > 0 \text{ such that } B_d(x, r) \subseteq U\},$$

where $B_d(x, r) = \{y \in X \mid d(x, y) < r\}$. This is: a set is open in the metric topology associated to d if for each of its points we can find an open ball around it (measured in the distance d , of course) which is still contained in U .

Let us show that the metric topology associated to $d_A := d|_{A \times A}$ is equal to the induced topology $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$.

Take an open set $V \in \mathcal{T}_A$. Then $V = U \cap A$ for some $U \in \mathcal{T}$. In order to show that V is open in \mathcal{T}_{d_A} , take any point $x \in V$. Since $U \in \mathcal{T} = \mathcal{T}_d$, there exists $r > 0$ such that

$B_d(x, r) \subseteq U$. Then $B_{d_A}(x, r) = B_d(x, r) \cap A \subseteq U \cap A = V$. This shows that V is open in the topology \mathcal{T}_{d_A} associated to the distance d_A , and hence that $\mathcal{T}_A \subseteq \mathcal{T}_{d_A}$.

Let us show now that $\mathcal{T}_{d_A} \subseteq \mathcal{T}_A$. using Exercise 3.14 (1) we have that $d : X \times X \rightarrow \mathbb{R}$ is continuous. Hence $d_A = d|_{A \times A}$ is also continuous (in the product topology of $A \times A$). By Exercise 3.14 (2), we have that $\mathcal{T}_{d_A} \subseteq \mathcal{T}_A$.