MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Problem sheet 3

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Unless otherwise specified, the symbols X, Y and Z represent topological spaces in the following exercises.

Exercise 3.1. (1 mark) Let $A \subseteq B \subseteq X$ be subsets of X. As usual, on B we consider the topology induced from X. Show that the topology on A induced from B is the same as the topology on A induced from X.

Exercise 3.2. A map $f : X \to Y$ is said to be *open* if $f(U)$ is open in Y for every U open in X. Show that the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open.

Exercise 3.3. (1 mark) Show that the countable collection

$$\{(a,b) \times (c,d) \mid a,b,c,d \in \mathbb{Q} \text{ and } a < b,\ c < d\}$$

is a basis for the usual topology on $\mathbb{R}^2$.

Exercise 3.4. If A, B are subsets of X, is it true in general that the following holds?

$$\overline{A} \setminus \overline{B} = \overline{A \setminus B}.$$

Exercise 3.5. Show that X is Hausdorff if and only if the diagonal $\Delta := \{(x,x) \mid x \in X\}$ is closed in $X \times X$.

Exercise 3.6. (1 mark) Show that for any subset A of X one has

$$\partial A = \overline{A} \cap \overline{X \setminus A}.$$

Exercise 3.7. Suppose that $f : X \to Y$ is continuous, and $A \subseteq X$. If x is a limit point of A, is it true that $f(x)$ is a limit point of $f(A)$?

Exercise 3.8. (1 mark) Show that a function $f : X \to Y$ is continuous if and only if

$$f^{-1}(\operatorname{int}(A)) \subseteq \operatorname{int}(f^{-1}(A))$$

for all sets $A \subseteq Y$.

Exercise 3.9. (1 mark) Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at exactly one point.

Exercise 3.10. Let $\{C_\alpha\}_{\alpha\in I}$ be a collection of closed subsets of a topological space X such that $\cup_{\alpha\in I} C_\alpha = X$. Take a function $f : X \to Y$ such that $f|_{C_\alpha}$ is continuous for each $\alpha \in I$. Assume that the family $\{C_\alpha\}_{\alpha\in I}$ is *locally finite*, this is: every point $x \in X$ has a neighbourhood that intersects C_α only for finitely many values of α. Show that f is continuous.

Exercise 3.11. (1 mark) Let $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ be continuous functions between topological spaces. Define

$$\begin{aligned} f \times g : X_1 \times X_2 &\to Y_1 \times Y_2 \\ (x_1, x_2) &\mapsto (f(x_1), g(x_2)). \end{aligned}$$

Show that $f \times g$ is continuous.

Exercise 3.12. (1 mark) Let $f : X \times Y \to Z$ be a continuous function. Show that f is continuous in each variable separately; this is: for each $x_0 \in X$, the function $h : Y \to Z$ defined by $h(y) = f(x_0, y)$ is continuous; and for each $y_0 \in Y$, the function $g : X \to Z$ defined by $h(x) = f(x, y_0)$ is continuous.

Is it true in general that if a function $f : X \times Y \to Z$ is continuous in each variable separately, then it is continuous?

Exercise 3.13. Let $A \subseteq X$ and $f : A \to Y$ be continuous. Assume that Y is Hausdorff. Assume also that it is possible to extend f to a continuous function $g : \overline{A} \to Y$. Show that there is only one possible such extension.

Exercise 3.14. (1 mark) Let X be a metric space with metric d.

1. Show that $d : X \times X \to \mathbb{R}$ is continuous.

2. Let $\mathscr{T}$ be a topology on X. Assume that $d : (X, \mathscr{T}) \times (X, \mathscr{T}) \to \mathbb{R}$ is continuous. Show that the topology $\mathscr{T}$ is finer than the metric topology of X.

Exercise 3.15. (1 mark) Let A be a subset of X. If d is a metric for the topology of X, show that $d|_{A\times A}$ is a metric for the induced topology on A.