MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Solutions to Problem Sheet 4

José A. Cañizo

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Unless otherwise specified, the symbols X, Y and Z represent topological spaces in the following exercises.

Exercise 4.1. This exercise suggests a way to show that a quotient space is homeomorphic to some other space. Consider an equivalence relation $\sim$ on X, and the quotient topological space $X^* \equiv X/\sim$. Let $f : X \to Y$ be a continuous and surjective map such that

1. For any $x, y \in X$ we have $f(x) = f(y)$ if and only if $x \sim y$.

2. f is open (this is, $f(U)$ is open whenever U is open) *or* f is closed (this is, $f(C)$ is closed whenever C is closed.)

Prove that the map $f^* : X^* \to Y$ given by $f^*(x^*) = f(x)$ for all $x \in X$ (where x^* represents the equivalence class of $x \in X$) is well defined, and is a homeomorphism.

Solution. In order to show that the map f^* is well defined we have to prove that whenever we have two points $x, y \in X$ such that $x^* = y^*$, it holds that $f(x) = f(y)$ (so that the definition of f^* is not ambiguous). Since $x^* = y^*$ is the same as $x \sim y$, this is implied by point 1 of the statement.

In addition, the map f^* is then injective also due to point 1, since if $f^*(x^*) = f^*(y^*)$ we have $x^* = y^*$. It is also surjective because f is, so f^* is bijective.

Let us check that f^* is continuous. If $U \subseteq Y$ is any set, we have

$$
\begin{aligned}
(f^*)^{-1}(U) = \{z \in X^* \mid f^*(z) \in U\} = \{x^* \mid x \in X \text{ and } f^*(x^*) \in U\} \\
= \{x^* \mid x \in X \text{ and } f(x) \in U\} = \pi(f^{-1}(U)),
\end{aligned}
$$

where $\pi : X \to X^*$ is the projection to the quotient. Hence, since the projection is always an open map and f is continuous, we see that $(f^*)^{-1}(U)$ is open.

Now, assume that f is open and let us check that the inverse map $(f^*)^{-1}$ is continuous. Equivalently, we may check that f^* is open. Take any open set $V \subseteq X^*$, which by definition of the topology of X^* must satisfy that $W := \pi^{-1}(V) \subseteq X$ is open. Since π is surjective we have $\pi(W) = \pi(\pi^{-1}(V)) = V$ (see Exercise 1.2 (6) from Problem Sheet 1). Hence,

$$f^*(V) = f^*(\pi(W)) = f(W).$$

Since f is open, this shows that f^* is open. Due to all of the above, f is a homeomorphism.

If f is closed we can follow the reasoning in the last paragraph to show that f^* is closed: take any closed set $C \subseteq X^*$. Since $X^* \setminus C$ is open, by definition of the topology of X^* it must happen that $\pi^{-1}(X^* \setminus C) = X \subseteq \pi^{-1}(C) \subseteq X$ is open. Hence $F := \pi^{-1}(C)$ is closed in X. For the same reason as above,

$$f^*(C) = f^*(\pi(F)) = f(F),$$

which is closed. This shows that f^* is closed; since it is bijective, this means that $(f^*)^{-1}$ is continuous, so in this case f^* is also a homeomorphism.

Exercise 4.2. Let X, Y be topological spaces with X compact and Y Hausdorff.

1. Prove that a continuous map $f : X \to Y$ must be closed.

2. Prove that a continuous bijection $f : X \to Y$ must be a homeomorphism.

Solution.

1. Take $C \subseteq X$ closed. Since X is compact, C must be compact (Lemma 9.6). Hence, $f(C)$ is compact (Theorem 9.8). Since Y is Hausdorff, $f(C)$ must be closed (Lemma 9.7).

2. It must be closed due to point 1. Since it is a bijection, its inverse is continuous (since if $C \subseteq X$ is closed we have $(f^{-1})^{-1}(C) = f(C)$, which is closed). Hence, it is a homeomorphism.

Exercise 4.3. Show rigorously that $[0, 1]/\{0, 1\}$ is homeomorphic to $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\} \subseteq \mathbb{R}^2$.

Solution. Recall that $[0, 1]/\{0, 1\}$ is the quotient topological space $[0, 1]/\sim$, where $\sim$ is the relation $x \sim y \Leftrightarrow (x = y \text{ or } \{x, y\} \subseteq \{0, 1\})$.

Consider the map $f : [0, 1] \to S^1$ given by $f(x) = (\cos(2\pi x), \sin(2\pi x))$, and let us show that it satisfies the conditions of Exercise 4.1: f is clearly continuous and surjective and we have that:

1. $f(x) = f(y)$ if and only if $x = y$ or $\{x, y\} \subseteq \{0, 1\}$, this is, if and only if $x \sim y$. Hence f satisfies point 1 of Exercise 4.1.

2. f is closed due to Exercise 4.2, since $[0, 1]$ is compact (Theorem 9.10) and S^1 is Hausdorff (due, for example, to Exercise 2.9 in Problem Sheet 2).

Then, Exercise 4.1 shows that f^* is a homeomorphism between $[0, 1]/\{0, 1\}$ and S^1.

Exercise 4.4. Show rigorously that the spaces Y referred to in Exercises 8.11 and 8.12 (Handout 8) are homeomorphic to $X/\mathscr{R}$.

Solution. (Sketch) One can apply the same kind of reasoning as in the previous exercise. In each case, one can find a map f from X to the candidate space which satisfies all requirements of Exercise 4.1.

Exercise 4.5. Show that in the finite complement topology of $\mathbb{R}$ (which we also called the cofinite topology), every subset of $\mathbb{R}$ is compact.

Solution. Take a nonempty set $A \subseteq \mathbb{R}$ and a cover $\mathscr{A}$ of A by open subsets of $\mathbb{R}$. The cover $\mathscr{A}$ must have at least a nonempty element U; by the definition of the cofinite topology, there is a (possibly empty) finite set of points $\{x_1, \ldots, x_N\} \subseteq \mathbb{R}$ such that $A \setminus U = \{x_1, \ldots, x_N\}$. Then, for each x_i ($i = 1, \ldots, N$) take $U_i \in \mathscr{A}$ with $x_i \in U_i$. The set

$$\tilde{\mathscr{A}} := \{U, U_1, \ldots, U_N\}$$

is clearly a finite subcover of $\mathscr{A}$ which covers A. Hence, A is compact.

Exercise 4.6. In the countable complement topology of $\mathbb{R}$ (which we also called the co-countable topology), is the subset $[0, 1]$ compact?

Solution. No. Define, for $n \in \mathbb{N}$,

$$U_n := \mathbb{R} \setminus \{1/k \mid k \geq n\}.$$

Then $\mathscr{A} = \{U_n \mid n \in \mathbb{N}\}$ is a cover of $[0, 1]$ by open sets which has no finite subcover of $[0, 1]$ (in order to cover $[0, 1]$ we must cover $1/k$ for all $k \in \mathbb{N}$, and hence for each $k \in \mathbb{N}$ we must include some U_n with $n \geq k$; this is not possible with a finite number of sets).

Exercise 4.7. Show that a finite union of compact subsets of X is compact.

Solution. Let $C_1, \ldots, C_n$ be compact sets, and let $\mathscr{A}$ be a cover of $C_1 \cup \cdots \cup C_n$. For any $i = 1, \ldots, n$, since C_i is compact there exists a finite set $\mathscr{A}_i \subseteq \mathscr{A}$ which covers C_i. Then $\mathscr{A}_1 \cup \cdots \cup \mathscr{A}_n$ is a finite subset of $\mathscr{A}$ which covers $C_1 \cup \cdots \cup C_n$.

Exercise 4.8.

1. Show that a compact subset of a metric space must be bounded.
2. Find a metric space in which not every closed and bounded subset is compact.

Solution.

1. Take any point x in the space. For any K compact, consider the cover $\{B(x, n) \mid n \in \mathbb{N}\}$. Since there is a finite subset of this that covers K, there must be $N \in \mathbb{N}$ with $K \subseteq B(x, N)$. Hence, K is bounded.
2. Consider the space $X = (0, 1)$, which is a metric space (with the usual distance). In it, the set $(0, 1)$ is closed (it is equal to the whole space X) and bounded, but $\{(1/n, 1) \mid n \in \mathbb{N}\}$ is an open cover of it which has no finite subcover.

Exercise 4.9. If Y is compact, show that the projection $\pi_X : X \times Y \to X$ is closed (this is, $\pi_X(C)$ is closed in X whenever C is closed in $X \times Y$.)

Solution. Take $C \subseteq X \times Y$ closed, and let us show that $\pi_X(C)$ is closed. Take a generic point $x \in X \setminus \pi_X(C)$, and let us show that there is an open set $U \subseteq X$ with $x \in U$ and $U \cap \pi_X(C) = \emptyset$. (This is enough to show that $\pi_X(C)$ is open.)

Take a closed set $C \subseteq X \times Y$. For any $y \in Y$ the point (x, y) is not in the closed set C, so we can find an open set of the form $U_y \times V_y$ with $U_y \subseteq X$ open, $V_y \subseteq Y$ open, $(x, y) \in U_y \times V_y$ and $(U_y \times V_y) \cap C = \emptyset$. Now, the collection $\{V_y \mid y \in Y\}$ covers Y, so using the compactness of Y we can find an open subcover $\{V_{y_1}, \ldots, V_{y_n}\}$. Now

$$U := \bigcap_{i=1}^{n} U_{y_i}$$

is an open set with $x \in U$ and $U \cap \pi_X(C) = \emptyset$.

Exercise 4.10. Consider $\mathbb{R}$ with the left limit topology.

1. Is the interval $[0, 1]$ compact?

2. Is the interval $[0, 1]$ connected?

Solution.

1. No. The cover $\{(-1, 0]\} \cup \{(1/n, 1] \mid n \in \mathbb{N}\}$ is an open cover of $[0, 1]$ which has no finite subcover.

2. No. It can be written as $[0, 1] = [0, 1/2] \cup (1/2, 1]$, which is a union of open sets in the induced topology on $[0, 1]$ (note that $[0, 1/2] = [0, 1] \cap (-1, 1/2]$, open in the induced topology on $[0, 1]$).

Exercise 4.11. A collection $\mathscr{C}$ of subsets of X is said to have the *finite intersection property* if every finite subfamily of $\mathscr{C}$ has nonempty intersection. Prove that the following are equivalent:

1. X is compact.

2. Every collection of closed sets in X having the finite intersection property satisfies that the intersection $\bigcap_{C \in \mathscr{C}} C$ of the whole family $\mathscr{C}$ is nonempty.

Solution. Assume that X is compact, and take a collection of closed sets $\mathscr{C}$ with the finite intersection property. Assume by contradiction that $\bigcap_{C \in \mathscr{C}} C = \emptyset$. Then $\mathscr{A} := \{X \setminus C \mid C \in \mathscr{C}\}$ is a collection of open sets such that

$$\bigcup_{U \in \mathscr{A}} U = \bigcup_{C \in \mathscr{C}} X \setminus C = X \setminus \bigcap_{C \in \mathscr{C}} C = X \setminus \emptyset = X,$$

this is, $\mathscr{A}$ is an open cover of X. Since X is compact, $\mathscr{A}$ contains a finite subcover of X, i.e., there are $C_1, \dots, C_n \in \mathscr{C}$ such that $\{U_1, \dots, U_n\} = \{X \setminus C_1, \dots, X \setminus C_n\}$ covers X. But then

$$\bigcap_{i=1}^{n} C_i = \bigcap_{i=1}^{n} X \setminus U_i = X \setminus \bigcup_{i=1}^{n} U_i = X \setminus X = \emptyset,$$

contradicting that $\mathscr{C}$ has the finite intersection property.

Conversely, assume point (2) of the statement. By contradiction, assume that X is not compact, so there is some open cover $\mathscr{A}$ of X with no finite subcover. By a similar argument as above, this means that the collection of closed sets $\mathscr{C} := \{X \setminus U \mid U \in \mathscr{A}\}$ has the finite intersection property. Hence its intersection cannot be empty, i.e.,

$$\emptyset \neq \bigcap_{C \in \mathscr{C}} C = \bigcap_{U \in \mathscr{A}} X \setminus U = X \setminus \bigcup_{U \in \mathscr{A}} U = X \setminus X = \emptyset,$$

a contradiction. Note that we used that $\mathscr{A}$ is a cover of X.