The lemma of de la Vallée-Poussin

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Here you will find a version of the classical lemma of de la Vallée-Poussin with a proof; a similar one can be found in [1].

**Proposition 1.** Let \( \mu \) be a positive Borel measure on \((0, +\infty)\), and \( f : (0, +\infty) \to \mathbb{R} \) a nonnegative \( \mu \)-integrable function. Then there is a measurable function \( \Phi : [0, +\infty) \to [0, +\infty) \) which is increasing, such that

\[
\lim_{y \to \infty} \Phi(y) = \infty,
\]

and

\[
\int_0^\infty \Phi f \mu < +\infty.
\]

In addition, the function \( \Phi \) can be chosen so that it is strictly increasing, \( \Phi(0) = 0 \), \( \Phi \) is \( C^\infty \), concave, and such that \( \Phi(y) \leq y \) for all \( y \geq 0 \).

If \( G : [0, +\infty) \to \mathbb{R} \) is a nonnegative function such that \( \lim_{y \to \infty} G(y) = +\infty \) and, for some \( \epsilon > 0 \) and all \( y \in [0, \epsilon] \), \( G(y) \geq \epsilon y \), then \( \Phi \) can be also chosen to be less than \( G \).

**Proof.** Define

\[
F(x) := \int_x^\infty f \mu
\]

which is a decreasing function and tends to zero as \( x \to \infty \) (as \( f \) is integrable). Define

\[
a_n := \inf \{ x > 0 \mid F(x) < 1/n^2 \} \in \mathbb{R}, \quad n \geq 1,
\]

and consider the increasing sequence \( \{x_n\}_{n \geq 0} \) given by

\[
x_0 := 0
\]

\[
x_{n+1} := \max \{x_n + 1, a_{n+1} + 1\}.
\]
The point of this sequence is that \( x_n \to \infty \) when \( n \to \infty \) (which is not necessarily true of \( a_n \)) and that

\[
F(x_n) \leq \frac{1}{n^2}.
\]

Finally, we can define \( \phi \):

\[
\chi_n := \chi_{[x_n,\infty)} \quad \text{for } n \geq 0
\]

\[
\phi := \sum_{n=0}^{\infty} \chi_n.
\]

The function \( \phi \) is well defined because for every \( x > 0 \), \( \phi(x) \) is given by a finite sum. Actually, we could define \( \phi \) equivalently as

\[
\phi(x) = n + 1 \quad \text{for } x \in [x_n, x_{n+1}), \quad n \geq 0.
\]

It is clear that \( \lim_{x \to \infty} \phi(x) = \infty \), as \( \phi(x) > n + 1 \) for \( x > x_n \). Also, the integral of \( \phi f \) is finite because

\[
\int_0^\infty \phi f \mu = \int_0^\infty \left( \sum_{n=0}^{\infty} \chi_n \right) f \mu = \sum_{n=0}^{\infty} \int_0^\infty \chi_n f \mu = \sum_{n=0}^{\infty} F(x_n) \leq \sum_{n=0}^{\infty} \frac{1}{n^2} < +\infty.
\]

(The monotone convergence theorem justifies the interchange of sums and integral here.)

Now, let us find a function \( \Phi \) in these conditions, which is also concave and strictly increasing, with \( \Phi(0) = 0 \) and \( \Phi(y) \leq y \) for \( y \geq 0 \). With the help of \( \phi \) and the above sequence \( \{x_n\} \), we will define \( \Phi \) recursively as follows:

\[
d_0 := 1;
\]

\[
\Phi(0) = 0;
\]

\[
d_{n+1} := \min \left\{ d_n, \frac{n + 1 - \Phi(x_n)}{x_{n+1} - x_n} \right\} \quad \text{for } n \geq 0
\]

\[
\Phi(x) := \Phi(x_n) + d_{n+1}(x - x_n) \quad \text{for } n \geq 0, \quad x \in [x_n, x_{n+1}].
\]

First, note that \( \Phi \) is continuous and \( \Phi(0) = 0 \) by definition. Its derivative on the interval \( (x_n, x_{n+1}) \) is \( d_{n+1} \); as \( \{d_n\} \) is decreasing and positive, \( \Phi \) is concave and strictly increasing, and as \( d_0 = 1 \), we have \( \Phi(y) \leq y \) for \( y \geq 0 \).
Figure 1: Definition of $\Phi$. The step function is $\phi$, and the piecewise linear one is $\Phi$. The scales on the axes are not the same.

Also, $\Phi(x)$ is smaller than $\phi(x)$, as for $x$ on the interval $[x_n, x_{n+1})$ ($n \geq 0$) one has

$$
\Phi(x) = \Phi(x_n) + d_{n+1}(x - x_n)
\leq \Phi(x_n) + \frac{n + 1 - \Phi(x_n)}{x_{n+1} - x_n}(x_{n+1} - x_n) = n + 1 = \phi(x).
$$

So the function $\Phi f$ is still $\mu$-integrable (as $\phi f$ is). Note that the latter inequality, written for $x = x_{n+1}$, also proves that $\Phi(x_n) \leq n$ for $n \geq 0$. Also, $\lim_{x \to \infty} \Phi(x) = \infty$. To prove this, observe that $d_n$ is always positive (as $\Phi(x_n) \leq n < n + 1$), so $\Phi$ is strictly increasing. Consider the set of the $n$ such that $d_{n+1}$ is different from $d_n$; if it is finite, then from some point on $\Phi$ has a constant positive slope and hence it tends to $\infty$; if it is infinite, then for all such $n$ one has

$$
\Phi(x_{n+1}) = \Phi(x_n) + d_{n+1}(x_{n+1} - x_n)
= \Phi(x_n) + \frac{n + 1 - \Phi(x_n)}{x_{n+1} - x_n}(x_{n+1} - x_n) = n + 1.
$$

(The equality holds because $d_{n+1}$ is not $d_n$, so it must be the other quantity in the minimum). So $\lim_{x \to \infty} \Phi(x) = \infty$. 

3
Now we can find a function $\Psi$ with the same properties as $\Phi$, and which is also $C^\infty$: extend $\Phi$ to all of $\mathbb{R}$ as

$$\Phi(x) := d_1 x \quad \text{for } x \leq 0.$$  

Take a “bump function” $\rho : \mathbb{R} \to \mathbb{R}$ which is $C^\infty$, nonnegative, with integral 1, symmetric about the $x = 0$ axis and with support contained in $[-1/2, 1/2]$.

The function

$$\Psi(x) := (\Phi * \rho)(x) = \int_{-\infty}^{\infty} \Phi(x-y) \rho(y) \, dy = \int_{-\infty}^{\infty} \Phi(y) \rho(x-y) \, dy$$

is the one we are looking for: $\Psi(0) = 0$, as $\Phi$ is equal to $d_1 y$ on the interval $[-1/2, 1/2]$ (recall that $x_1 \geq 1$) and $\rho$ is symmetric, so

$$\Psi(0) = \int_{-\infty}^{\infty} \Phi(y) \rho(-y) \, dy = d_1 \int_{-1/2}^{1/2} y \rho(-y) \, dy = 0.$$  

$\Psi$ is $C^\infty$, being a regularization of $\Phi$ by a $C^\infty$ function; it is less than $x$, as for $0 \leq x \leq 1/2$ we know that $\Psi(x) = d_1 x \leq x$, and for $x \geq 1/2$ we have, using the symmetry of $\rho$ and the bound for $\Phi$,

$$\Psi(x) = \int_{-\infty}^{\infty} \rho(y) \Phi(x-y) \, dy \leq \int_{-\infty}^{\infty} \rho(y) (x-y) \, dy$$

$$= x \int_{-\infty}^{\infty} \rho(y) \, dy - \int_{-\infty}^{\infty} \rho(y) y \, dy = x.$$  

(Note that $\Phi(x)$ is not less than $x$ for $x < 0$, so this calculation does not work for $0 \leq x < 1/2$). $\Psi$ is concave and strictly increasing because $\Phi$ is, and convolution with a positive function preserves this; $\Psi(x)$ tends to $\infty$ when $x \to \infty$, and if we observe that for $x \geq 0$

$$\Psi(x) = \int_{-\infty}^{\infty} \Phi(y) \rho(x-y) \, dy \leq \|\rho\|_\infty \Phi(x+1/2) \leq \|\rho\|_\infty (\Phi(x) + \Phi(1/2)), \quad (1)$$

(note that $\Psi$ is sublinear, as it is concave and $\Psi(0) = 0$, so $\Psi(x + y) \leq \Psi(x) + \Psi(y)$ for $x, y \geq 0$), then it is clear that $\Psi f$ is integrable on $(0, +\infty)$.

Finally, let us see that $\Psi$ can be chosen to be less than a $G$ in the conditions of the statement. Call

$$b_n := \inf\{x \in [0, +\infty) \mid G(x) > n + 1\} < +\infty.$$
In the definition at the beginning of the proof, put \( y_n := \max x_n, b_n + 1 \), and define \( \phi \) using \( y_n \) instead of \( x_n \). Then,

\[
\phi(x) \leq G(x) + 1 \quad \text{for } x \geq x_1.
\]

Define \( \Phi \) accordingly (so \( \Phi(x) \leq G(x) + 1 \) for \( x \geq x_1 \)), and choose \( \delta > 0 \) such that

\[
\delta \leq \min\{1, 1/\|\rho\|_{\infty}, 1/(\|\rho\|_{\infty} \Phi(1/2))\}.
\]

Then define \( \Psi \) as the convolution above, times \( \delta \):

\[
\Psi := \delta \Phi \ast \rho.
\]

The bound in (1) proves that \( \Psi(y) \leq G(y) \) for \( y \geq x_1 \), and this \( \Psi \) still satisfies all the other properties of the proposition. Now we only have to choose another \( \delta > 0 \) such that

\[
\delta \Psi'(0) \leq \epsilon \\
\delta \Psi(x) \leq G(x) \quad \text{for } \epsilon \leq x \leq x_1,
\]

and then \( \delta \Psi \) is less than \( G \) (recall that \( G(x) \geq \epsilon x \) for \( x \in [0, \epsilon] \) and \( \Psi \) is concave) and satisfies all the other properties. \( \square \)

In the rest of this section, \( S \) will be a set, \( \mathcal{A} \) will be a \( \sigma \)-algebra of subsets of \( S \) and \( \mu \) be a positive measure on \( \mathcal{A} \).

**Proposition 2.** Consider the positive measure space \((S, \mathcal{A}, \mu)\). If \( f : S \to \mathbb{R} \) is a nonnegative \( \mu \)-integrable function, then there is a continuous function \( \Lambda : [0, +\infty) \to [0, +\infty) \) which is increasing, such that \( \lim_{y \to \infty} \Lambda(y)/y = \infty \), and

\[
\int_0^\infty \Lambda(f(y))\mu(y) < +\infty.
\]

The function \( \Lambda \) can be chosen so that \( \Lambda(0) = 0 \), \( \Lambda \) is \( C^\infty \), and strictly convex.

If \( H : [0, +\infty) \to \mathbb{R} \) is an absolutely continuous function so that \( G = H' \) is in the conditions of \( G \) in proposition 1, then \( \Lambda \) can be chosen to be less than \( H \).

This result is a corollary of the previous proposition if one uses the concept of the *distribution function* of a given function \( f \):
Definition 3. If \( f : S \to \mathbb{R} \) is a nonnegative \( \mu \)-integrable function, then its distribution function is the function \( F_f : (0, +\infty) \to [0, +\infty) \) given by

\[
F_f(\lambda) := \mu\{ y \in X \mid f(y) > \lambda \} \quad \text{for} \quad \lambda > 0.
\]

Note that the set \( \{ y \in X \mid f(y) > \lambda \} \) is measurable, as \( f \) is. It is clear that \( F_f \) is decreasing, so in particular it is Borel measurable. The following lemma gives a way to calculate the integral of \( \varphi(f) \) for suitable functions \( \varphi \) knowing only the distribution function \( F_f \).

Lemma 4. Let \( \varphi : [0, +\infty) \to [0, +\infty) \) be a nonnegative \( \mathcal{C}^1 \) function such that \( \varphi(0) = 0 \), and \( f : S \to \mathbb{R} \) a nonnegative \( \mu \)-integrable function. Then

\[
\int_S \varphi(f(x))\mu(x) = \int_0^\infty F_f(\lambda)\varphi'(\lambda)\,d\lambda.
\]

Proof. To prove this, note first that the function

\[
G : S \times [0, +\infty) \to \mathbb{R},
\]

\[
(x, t) \mapsto f(x) - t
\]

is measurable for the product \( \sigma \)-algebra \( \mathcal{A} \otimes \mathcal{B} \), as it is a sum of two measurable functions. Hence, the set \( \{(x, t) \in S \times [0, +\infty) \mid f(x) < t\} \) is measurable, and therefore the function

\[
\chi : S \times [0, +\infty) \to \mathbb{R},
\]

\[
(x, t) \mapsto \begin{cases} 1 & \text{if } f(x) < t \\ 0 & \text{if } f(x) \geq t \end{cases}
\]

is measurable. Observe that

\[
F_f(\lambda) = \int_S \chi(x, \lambda)\mu(x) \quad \text{for} \quad \lambda > 0.
\]

Hence we can apply Fubini’s theorem and write

\[
\int_0^\infty F_f(\lambda)\varphi'(\lambda)\,d\lambda = \int_0^\infty \int_S \chi(x, \lambda)\mu(x)\varphi'(\lambda)\,d\lambda
\]

\[
= \int_S \int_0^\infty \chi(x, \lambda)\varphi'(\lambda)\,d\lambda\mu(x) = \int_S \int_0^{f(x)} \varphi'(\lambda)\,d\lambda\mu(x) = \int_S \varphi(f(x))\mu(x).
\]

This proves the lemma. \( \square \)
Now we can prove proposition 2:

**Proof of proposition 2.** The previous lemma proves that \( \int_S f \mu = \int_0^\infty F_f(\lambda) \, d\lambda \), so \( F_f \) is integrable. Proposition 1 then shows that there is a \( C^\infty \) nonnegative concave function on \([0, +\infty)\), which we call \( \Lambda' \), such that \( \Lambda'(0) = 0 \), \( \lim_{\lambda \to \infty} \Lambda'(\lambda) = +\infty \) and

\[
\int_0^\infty F_f(\lambda) \Lambda'(\lambda) \, d\lambda < +\infty.
\]

We define \( \Lambda \) as its primitive:

\[
\Lambda(\lambda) := \int_0^\lambda \Lambda'(y) \, dy.
\]

Then \( \Lambda \) clearly fulfills the requirements of the proposition; in particular,

\[
\int_S \Lambda(f(x)) \mu(x) = \int_0^\infty F_f(\lambda) \Lambda'(\lambda) \, d\lambda < +\infty,
\]

and also, using l’Hôpital’s rule,

\[
\lim_{\lambda \to \infty} \frac{\Lambda(\lambda)}{\lambda} = \lim_{\lambda \to \infty} \frac{\Lambda'(\lambda)}{1} = +\infty.
\]

Finally, if \( H \) is in the conditions of the proposition, we may choose \( \Lambda' \) less than \( H' \) and the result follows. \( \square \)

# 1 About this text

This document has been written by José Alfredo Cañizo. For comments or suggestions write to ozarfreo@yahoo.com. The latest version should be at http://www.ugr.es/~ozarfreo/tex.

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## References