Nonlocal interaction equations with singular kernels: Wasserstein gradient flow vs entropy solutions

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Non local interaction PDE

$$\mu \in \mathcal{P}_2(\mathbb{R}),\ W(x) = -|x| \text{ or } W(x) = |x|$$

$$\frac{\partial \mu}{\partial t} = \frac{\partial}{\partial x}\left(\mu \frac{\partial}{\partial x} W * \mu\right), \qquad x \in \mathbb{R}, \quad t > 0,$$

$$\mathcal{W}(\mu) = \frac{1}{2}\int W(x-y)d\mu d\mu$$

Burgers equation

$$F \in L^\infty(\mathbb{R}; [0,1]),\ \partial_x F \in \mathcal{P}_2(\mathbb{R})$$

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial x} g(F) = 0, \qquad x \in \mathbb{R}, t > 0.$$

$$\text{where } g(F) = \pm(F^2 - F).$$

ODE in L^2

$$X \in \mathcal{K} := \{f \in L^2([0,1]) \mid f \text{ is non-decreasing}\}$$

$$\partial_t X_t(s) = h(s), \qquad s \in [0,1], \quad t > 0,$$

$$h(s) = \begin{cases} \int_0^1 \operatorname{sign}(X_t(z) - X_t(s))dz \\ 2s - 1 \end{cases}$$

What we address

- Equivalence of the three systems
- Approximation of solutions
- Characterization of the subdifferential of $\mathcal{W}$

Equivalence

The variables μ, F, X are mutually defined

$$\mu \mapsto F_\mu(x) \mapsto X_\mu(s)$$

$$F_\mu = \mu((-\infty, x])$$

$$X_\mu(s) = \inf\{x | F_\mu(x) > s\}$$

and the relation between the equations

$$\frac{\partial}{\partial x}\left[\partial_t F \pm \partial_x(F^2 - F)\right] = 0$$

$$\downarrow$$

$$\partial_t \mu = \pm \partial_x (\mu(2F - 1))$$

$$\downarrow$$

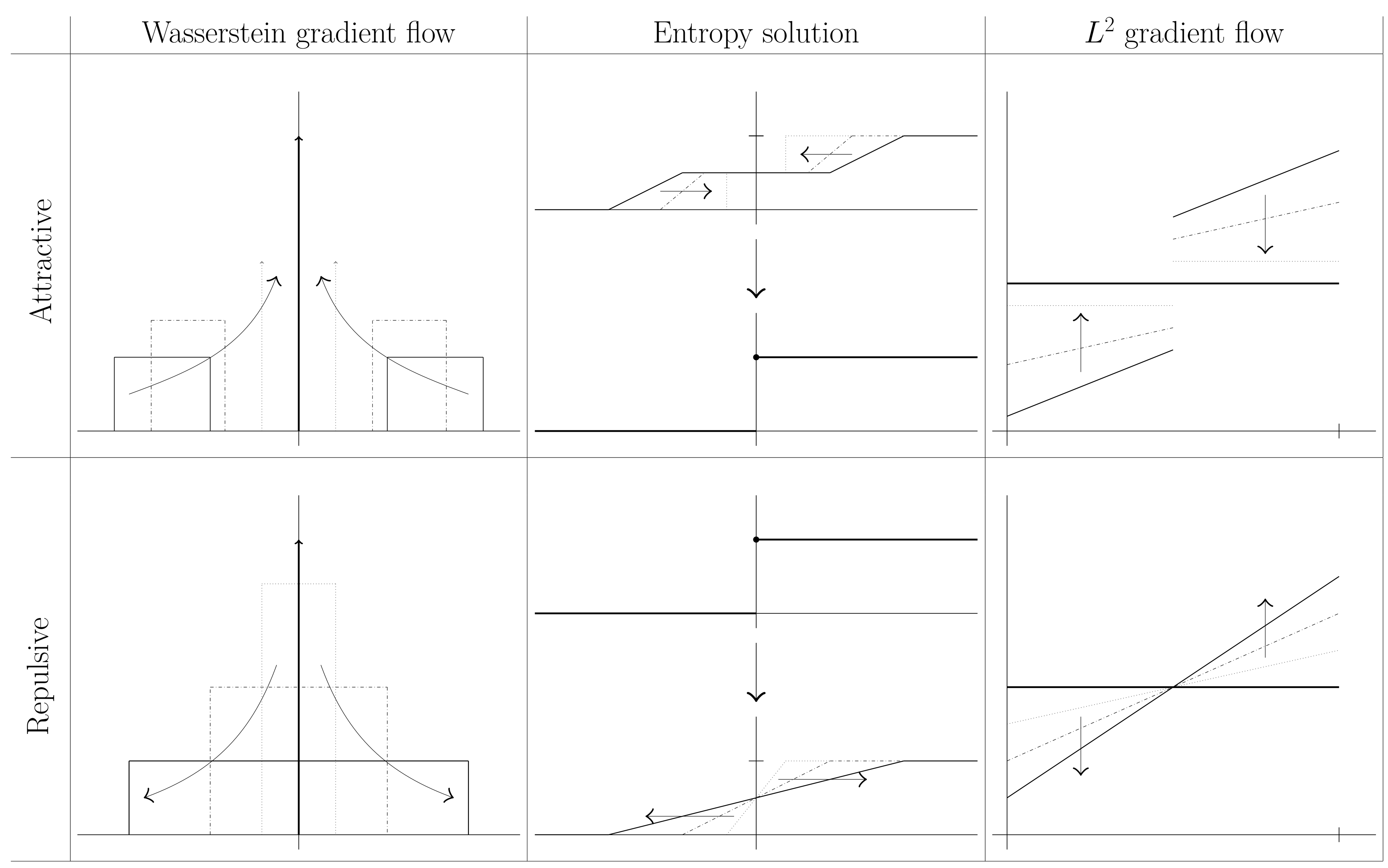
$$\pm(2F - 1) = v = -\partial \mathcal{W},$$

$$\partial_t \mu + \partial_x(\mu v) = 0$$

$$\downarrow$$

$$\mathcal{W}(\mu) = \frac{1}{2}\int \pm|X(s) - X(z)|dsdz = \int \pm X(s)(2s-1)ds$$

Example of evolution for the three systems



Theorem (Equivalence among Wasserstein gradient flows, L^2 gradient flows, and entropy solutions.)

Let $W(x) = \sigma|x|$ with $\sigma \in \{-1, 1\}$. Let $\mu_0 \in \mathcal{P}(\mathbb{R})$. Let $F_0(x) = \mu_0((-\infty, x])$ and X_0 be the pseudo-inverse of F_0. Let $g(F) = \sigma(F^2 - F)$. Let $\mu_t \in AC([0, +\infty)) \to \mathcal{P}_2(\mathbb{R})$. Then, the following are equivalent:

1 *The curve μ_t is the unique Wasserstein gradient flow solution with initial condition μ_0.*

2 *The curve $F(x,t) = \mu_t((-\infty, x])$ is the unique entropy solution to the scalar conservation law with initial condition F_0.*

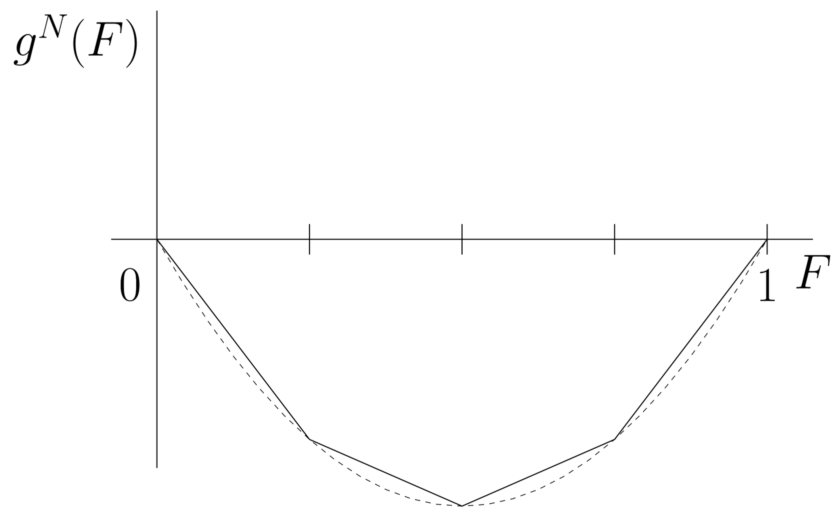
3 *The curve $X_t(s) = \inf\{x \mid F(x,t) > s\}$ is the unique L^2 gradient flow with initial condition X_0.*

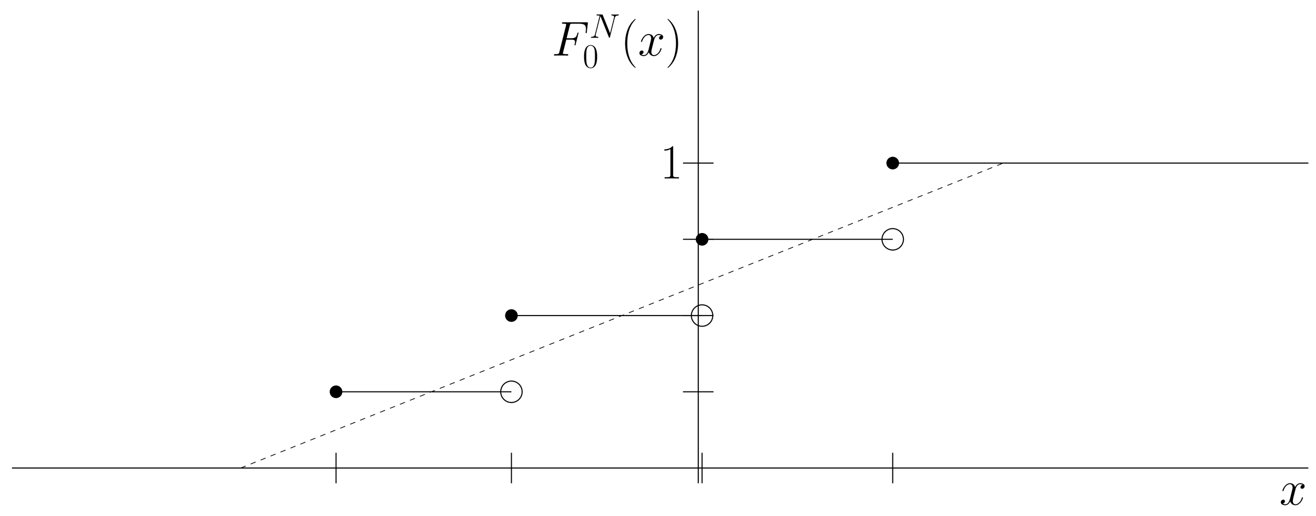
Theorem (Particle approximation in the repulsive case)

Let $\mu_0 \in \mathcal{P}_2(\mathbb{R})$, and let $\mu(x,t)$ be the unique gradient flow solution of $\mathcal{W}$ with $\sigma = -1$ with initial datum μ_0. For each N, let μ^N be the empirical measure $\mu^N(t) = \sum_{j=1}^N \frac{1}{N}\delta_{x_j^N(t)}$ with x_j^N satisfying

$$\dot{x}_j^N(t) = \frac{1}{N}\sum_{k \neq j} \operatorname{sign}(x_j^N(t) - x_k^N(t)) = \frac{2j - 1 - N}{N}, \qquad x_j^N(0) = X_0(j/N), \qquad j = 1, ..., N.$$

Then, for all $t \geq 0$, we have $\lim_{N\to\infty} d_W(\mu^N(t), \mu(t)) = 0$.





Theorem (Characterization of the subdifferential)

Given the functional $\mathcal{W}$ and a measure $\mathcal{P}_2(\mathbb{R}) \ni \mu_0 = \nu + \sum_{i\in I} m_i \delta_{x_i}$, for some finite or countable I and with $\nu(\{x\}) = 0$ for every $x \in \mathbb{R}$, then, defining α_i such that $X_{\mu_0} = x_i$ on $(\alpha_i, \alpha_i + m_i)$, $\Delta_i = [2\alpha_i - 1, 2(\alpha_i + m_i) - 1]$, $\mathcal{X}_{\Delta_i}$ the characteristic function of the interval Δ_i and $k_0(x) := 2F_0(x) - 1$, the plan

$$\boldsymbol{\gamma}(x,y) = \sum_i \frac{1}{2}\delta_{x_i} \otimes \mathcal{X}_{\Delta_i} + (i \otimes k_0)_{\#}\nu,$$

is the unique element of minimal norm in $\boldsymbol{\partial}\mathcal{W}(\mu_0)$.

Outlook

- Connection for general W
- Non-convex interaction in $\mathbb{R}^n$, $n > 1$
- Different framework as in Gigli-Otto *"Entropic Burgers equation via a minimizing movement scheme based on the Wasserstein metric."* CVPDE (2012)

References

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