Mean-field limit and propagation of chaos for aggregation equations

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Outline

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Introduction

- Ø Mean-field limit for aggregation equations
- O Local existence and uniqueness of solutions
- Propagation of chaos

Swarming in nature



What is the swarming?

"Swarming" is a collective behavior exhibits by agents of similar size and body type moving in a coordinated way.

- Describing collective behaviours in nature; insects(ants, bees, ...), fishes, birds, micro-organisms(myxo-bacteria).
- Industry; formation controls of robots, unmanned aerial vehicles, etc.
- 3 interaction regions:



1st order aggregation equations: motivation

Consider the Newton's equations with very small variations of the velocity and speed¹:

$$m\frac{d^2x_i}{dt^2} + \alpha\frac{dx_i}{dt} + \sum_{j\neq i}\nabla W(|x_i - x_j|) = 0.$$

Then we can formally derive 1st order particle and its continuum equations:

$$\frac{dx_i}{dt} = \sum_{j \neq i} \nabla W(|x_i - x_j|) \text{ mean field limit}(N \to \infty) \Rightarrow \begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0\\ u = -\nabla W * \rho. \end{cases}$$

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¹Edelshtein-Keshet and Mogilner(1999)

Mathematical tools: Wasserstein distance

Definition 1. (Wasserstein *p*-distance)

Let ρ_1 , ρ_2 be two Borel probability measures on \mathbb{R}^d . Then the Euclidean Wasserstein distance of order $1 \leq p < \infty$ between ρ_1 and ρ_2 is defined as

$$d_p(
ho_1,
ho_2):=\inf_{\gamma}\left(\int_{\mathbb{R}^d imes\mathbb{R}^d}\left|x-y
ight|^pd\gamma(x,y)
ight)^{1/p},$$

where the transference plan γ runs over the set of joint probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals ρ_1 and $\rho_2 \in \mathcal{P}_p(\mathbb{R}^d)$. For $p = \infty$ (this is the limiting case, as $p \to \infty$),

$$d_{\infty}(
ho_1,
ho_2):=\inf_{\gamma}\left(\sup_{(x,y)\in\mathsf{supp}(\gamma)}|x-y|
ight),$$

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Definition 2.

Let ρ_1 be a Borel measure on \mathbb{R}^d and $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable mapping. Then the push-forward of ρ_1 by \mathcal{T} is the measure ρ_2 defined by

 $\rho_2(B) = \rho_1(\mathcal{T}^{-1}(B)) \quad \text{for} \quad B \subset \mathbb{R}^d,$

and denoted as $\rho_2 = \mathcal{T} \# \rho_1$.

Remark 1.

The definition of $\rho_2 = \mathcal{T} \# \rho_1$ is equivalent to

$$\int_{\mathbb{R}^d} \phi(x) \, d\rho_2(x) = \int_{\mathbb{R}^d} \phi(\mathcal{T}(x)) \, d\rho_1(x) \,,$$

for all $\phi \in \mathcal{C}_b(\mathbb{R}^d)$. Given a probability measure with bounded *p*-th moment ρ_0 , consider two measurable mappings $X_1, X_2 : \mathbb{R}^d \to \mathbb{R}^d$, then the following inequality holds.

$$d_p^p(X_1 \# \rho_0, X_2 \# \rho_0) \le \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\gamma(x, y) = \int_{\mathbb{R}^d} |X_1(x) - X_2(x)|^p d\rho_0(x).$$

Here, we used as transference plan $\gamma = (X_1 \times X_2) \# \rho_0$.

Aggregation equation

This model consists of the continuity equation for the probability density of individuals $\rho(t, x)$ at position $x \in \mathbb{R}^d$ and time t > 0 given by:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, & t > 0, \quad x \in \mathbb{R}^d, \\ u(t, x) := -\nabla W * \rho, & t > 0, \quad x \in \mathbb{R}^d, \\ \rho(0, x) := \rho^0(x), & x \in \mathbb{R}^d, \end{cases}$$
(1)

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where u(t,x) is velocity field non-locally computed in terms of the density of individuals.

Approximation by particles

As an approximation by particles of the aggregation equations (1), we consider the following ODE system:

$$\begin{cases} \dot{X}_{i}(t) = -\sum_{j \neq i} m_{j} \nabla W(X_{i}(t) - X_{j}(t)), \\ X_{i}(0) = X_{i}^{0}, \quad i = 1, \dots, N. \end{cases}$$
(2)

Here, $\{X_i\}_{i=1}^N$ and $\{m_i\}_{i=1}^N$ are the positions and weights of *i*-th particles, respectively. We define the associated empirical distribution $\mu_N(t)$ as

$$\mu_N(t) = \sum_{i=1}^N m_i \delta_{X_i(t)}, \quad \sum_{i=1}^N m_i = \int_{\mathbb{R}^d} \rho_0(x) dx = 1,$$
(3)

with $m_i > 0$, i = 1, ..., N. We set $\nabla W(0) = 0$ even if there is a singular at the origin.

A question on the mean-field limit

As long as two particles (or more) do not collide, μ_N satisfies (1) in the sense of distributions, i.e., $\mu_N(t)$ and $\rho(t)$ satisfy the same equation. In this framework, the convergence:

" $\mu_N^0 \rightharpoonup \rho^0$ weakly-* as measures $\Longrightarrow \mu_N(t) \rightharpoonup \rho(t)$ weakly-* as measures for small time or for every time?"

is a natural question.

Notations

• Quantities to estimate: d_{∞} -distance between $\rho(t)$ and $\mu_N(t)$ and minimum inter-particle distance:

 $\eta(t) := d_{\infty}(\mu_N(t), \rho(t)), \quad \eta_m(t) := \min_{1 \le i \ne j \le N} \left(|X_i(t) - X_j(t)| \right), \quad (4)$ with $\eta^0 := \eta(0)$ and $\eta^0_m := \eta_m(0).$

• Functional space: Solutions of the aggregation equations (1) in $L^{\infty}(0,T;(L^1 \cap L^p)(\mathbb{R}^d))$ with $1 \leq p \leq \infty$ to be determined depending not he singularity of the potential. We set

 $\|\rho\|_{(L^1 \cap L^p)(\mathbb{R}^d)} := \|\rho\|_1 + \|\rho\|_p, \quad \|\rho\| := \|\rho\|_{L^{\infty}(0,T;(L^1 \cap L^p)(\mathbb{R}^d))},$

where $\|\rho\|_p$ denotes the $L^p(\mathbb{R}^d)$ -norm of ρ , $1 \leq p \leq \infty$.

Assumption on the potential function W(x)

In order to make sense of solutions to (1), we need the following assumptions on the interaction potential: we first fix $\nabla W(0) = 0$ by definition, and

$$|\nabla W(x)| \le \frac{C}{|x|^{\alpha}}, \quad \text{and} \quad |D^2 W(x)| \le \frac{C}{|x|^{1+\alpha}}, \quad \forall \ x \in \mathbb{R}^d \setminus \{0\},$$
 (5)

for $0 \leq \alpha < d-1$. Note that due to the assumptions on W, we can always find $1 such that <math>(\alpha + 1)p' < d$, and thus ∇W belongs to $\mathcal{W}_{loc}^{1,p'}(\mathbb{R}^d)$.

We remark that our strategy does not take advantage of the repulsive or attractive character of the potentials.

Statement of the mean field limit

Theorem 1. (Mean field limit)

Suppose the kernel W satisfies (5), and let ρ be a solution to the system (1) up to time T > 0, such that $\rho \in L^{\infty}(0,T; (L^1 \cap L^p)(\mathbb{R}^d)) \cap \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d))$, with initial data $\rho^0 \in (\mathcal{P}_1 \cap L^p)(\mathbb{R}^d)$, $0 \le \alpha < -1 + d/p'$, and 1 . $Furthermore, we assume <math>\mu_N^0$ converges to ρ^0 for the distance d_{∞} as the number of particles N goes to infinity, i.e.,

 $d_\infty(\mu^0_N,
ho^0) o 0$ as $N o\infty,$

and that the initial quantities $\eta^0, \ \eta^0_m$ satisfy

$$\lim_{N \to \infty} \frac{(\eta^0)^{d/p'}}{(\eta^0_m)^{1+\alpha}} = 0.$$
 (6)

Theorem 1. (Continued)

Then, for N large enough the particle system (2) is well-defined up to time T, in the sense that there is no collision between particles before that time, and moreover

 $\mu_N(t) \rightharpoonup \rho(t) \quad \text{weakly-* as measures as} \quad N \to \infty, \quad \text{for all} \quad t \in [0,T].$

Strategy of the proof

• In Step A, we estimate the growth of the d_{∞} Wasserstein distance between the continuum and the discrete solutions η that involves η itself and η_m in the form:

$$\frac{l\eta}{lt} \le C\eta \|\rho\| \left(1 + \eta^{d/p'} \eta_m^{-(1+\alpha)}\right).$$
(7)

• In Step B, we estimate the decay of the minimum inter-particle distance η_m , which also involves the terms η and η_m in the form:

$$\frac{d\eta_m}{dt} \ge -C\eta_m \|\rho\| \left(1 + \eta^{d/p'} \eta_m^{-(1+\alpha)}\right).$$
(8)

• In Step C, under the assumption of the initial approximation (6), we combine (7) and (8) to conclude the desired result.

Step A: Estimate for growth of the $d_{\infty}(t)$

We first define the flows $\Psi_N, \Psi: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ as solutions of

$$\begin{cases} \frac{d}{dt}(\Psi(t;s,x)) = u(t;s,\Psi(t;s,x)),\\ \Psi(s;s,x) = x, \end{cases}$$
(9)

for all $s, t \in [0, T]$, and

$$\begin{cases} \frac{d}{dt}(\Psi_N(t;s,x)) = u_N(t;s,\Psi_N(t;s,x)),\\ \Psi_N(s;s,x) = x, \end{cases}$$
(10)

for all $s, t \in [0, T_0^N]$. Here $u(x, t) = -\nabla W * \rho$ and $u_N := -\nabla W * \mu_N$. We can easily check that the flow map $\Psi_N(t; s, x)$ solution to (10) is well-defined for $t, s \in [0, T_0^N]$.

Mean-field limit and propagation of chaos for aggregation equations Mean-field limit for the aggregation equation

Assumptions (5) imply that

$$\nabla W(x) - \nabla W(y)| \le \frac{2|x-y|}{\min(|x|, |y|)^{\alpha+1}}.$$
(11)

The estimate (11) implies that the velocity field is Lipschitz continuous with respect to the spatial variable. Actually, one can estimate it as

$$\begin{split} |u(t,x) - u(t,y)| &\leq \int_{\mathbb{R}^d} |\nabla W(x-z) - \nabla W(y-z)| \rho(t,z) \, dz \\ &\leq 2|x-y| \int_{\mathbb{R}^d} \frac{1}{\min(|x-z|,|y-z|)^{\alpha+1}} \rho(t,z) \, dz \\ &\leq 4|x-y| \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-z|^{\alpha+1}} \rho(t,z) \, dz \, . \end{split}$$

Now, splitting the last integral into the near- and far-field sets $\mathcal{A} := \{z : |x - z| \ge 1\}$ and $\mathcal{B} := \mathbb{R}^d - \mathcal{A}$ and estimating the two terms, we deduce

$$\int_{\mathbb{R}^d} \frac{1}{|x-z|^{\alpha+1}} \rho(t,z) \, dz \le \|\rho(t)\|_1 + \left(\int_{\mathcal{B}} \frac{1}{|x-y|^{(1+\alpha)p'}} \, dy\right)^{1/p'} \|\rho(t)\|_p \le C \|\rho\|,$$
(12)

for all $x\in \mathbb{R}^d$ due to the assumption $(1+\alpha)p' < d.$ The second s Fixed $0 \le t_0 < \min(T, T_0^N)$ and choose an optimal transport map for d_∞ denoted by \mathcal{T}^0 between $\rho(t_0)$ and $\mu_N(t_0)$; $\mu_N(t_0) = \mathcal{T}^0 \# \rho(t_0)$. Then $\rho(t) = \Psi(t; t_0, \cdot) \# \rho(t_0)$ and obviously $\mu_N(t) = \Psi_N(t; t_0, \cdot) \# \mu_N(t_0)$ for $t \ge t_0$. We also notice that for $t \ge t_0$

$$\mathcal{T}^t \#
ho(t) = \mu_N(t), \quad ext{where} \quad \mathcal{T}^t = \Psi_N(t;t_0,\cdot) \circ \mathcal{T}^0 \circ \Psi(t_0;t,\cdot).$$

It follows from the property of Wasserstein distance that

$$\eta(t) = d_{\infty}(\mu_N(t), \rho(t)) \le \|\Psi(t; t_0, \cdot) - \Psi_N(t; t_0, \cdot) \circ \mathcal{T}^0\|_{\infty}.$$

We notice that

$$rac{d}{dt}\left(\Psi_N(t;t_0,\mathcal{T}^0(x))-\Psi(t;t_0,x)
ight)\Big|_{t=t_0}=u_N(t_0,\mathcal{T}^0(x))-u(t_0,x).$$

We also find

$$\frac{d}{dt} \|\Psi_N(t;t_0,\cdot) \circ \mathcal{T}^0 - \Psi(t;t_0,\cdot)\|_{\infty}\Big|_{t=t_0^+} \le \|u_N(t_0,\cdot) \circ \mathcal{T}^0 - u(t_0,\cdot)\|_{\infty}.$$

We notice that

$$u_N(t_0, \mathcal{T}^0(x)) - u(t_0, x)$$

= $-\int_{\mathbb{R}^d} \nabla W(\mathcal{T}^0(x) - y) d\mu_N(t_0, y) + \int_{\mathbb{R}^d} \nabla W(x - y) \rho(t_0, y) dy$
= $-\int_{\mathbb{R}^d} \left(\nabla W(\mathcal{T}^0(x) - \mathcal{T}^0(y)) - \nabla W(x - y) \right) \rho(t_0, y) dy.$

For notational simplicity, we omit the time dependency on t_0 in the next few computations. This yields

$$\frac{d^{+}\eta}{dt} \leq C \sup_{x \in \mathbb{R}^{d}} \int_{\mathbb{R}^{d}} |\nabla W(\mathcal{T}(x) - \mathcal{T}(y)) - \nabla W(x - y)|\rho(y)dy.$$
(13)

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We decompose the integral on \mathbb{R}^d into the near- and the far-field parts as $\mathcal{A} := \{z : |x - z| \ge 4\eta\} \text{ and } \mathcal{B} := \mathbb{R}^d - \mathcal{A} \text{ as}$ $\int_{\mathbb{R}^d} |\nabla W(\mathcal{T}(x) - \mathcal{T}(y)) - \nabla W(x - y)|\rho(y)dy = \int_{\mathcal{A}} \cdots + \int_{\mathcal{B}} \cdots$ $:= \mathcal{I}_1 + \mathcal{I}_2.$ (14)

For the estimate in the set \mathcal{A} , we use (11) and (12) to obtain

$$\mathcal{I}_1 \leq \int_{\mathcal{A}} \frac{2\left(|x - \mathcal{T}(x)| + |y - \mathcal{T}(y)|\right)}{\min(|x - y|, |\mathcal{T}(x) - \mathcal{T}(y)|)^{\alpha + 1}} \rho(y) dy \leq C\eta \|\rho\|.$$

For the second part I_2 , we estimate separately each term using (5) to deduce

$$\mathcal{I}_{2} \leq \int_{\mathcal{B}} \frac{\rho(y)}{|x-y|^{\alpha}} dy + \int_{\mathcal{B}} \frac{\rho(y)}{\eta_{m}^{\alpha}} dy \\
\leq C(\eta^{d/p'-\alpha} + \eta^{d/p'} \eta_{m}^{-\alpha}) \|\rho\|_{p} \leq C(\eta^{d/p'-\alpha} + \eta^{d/p'} \eta_{m}^{-\alpha}) \|\rho\|.$$
(15)

Hence we have

$$\frac{d^+\eta}{dt} \le C\eta \|\rho\| \left(1 + \eta^{d/p'-1}\eta_m^{-\alpha}\right) \le C\eta \|\rho\| \left(1 + \eta^{d/p'}\eta_m^{-(1+\alpha)}\right), \ \forall t \in [0,\min(T,T_0^N)]$$

where we used $\eta_m \le 2\eta$.

Step B: Estimate for decay of the $\eta_m(t)$

We choose two indices i, j so that $|X_i - X_j| = \eta_m$. Then we get

$$\begin{aligned} \frac{d}{dt} |X_i - X_j| &\geq -|u_N(X_i) - u_N(X_j)| \\ &\geq -\int_{\mathbb{R}^d} |\nabla W(X_i - y) - \nabla W(X_j - y)| \, d\mu_N(y) \\ &= -\int_{\mathbb{R}^d} |\nabla W(X_i - \mathcal{T}(y)) - \nabla W(X_j - \mathcal{T}(y))| \, \rho(y) dy \,, \end{aligned}$$

where we used $\mu_N(t) = \mathcal{T} \# \rho(t)$, for each $t \in [0, \min(T, T_0^N))$. Similar to (14), we split in near- and far-field parts the domain \mathbb{R}^d as $\mathcal{A} := \{y : |X_i - y| \ge 2\eta \text{ and } |X_j - y| \ge 2\eta\}$ and $\mathcal{B} := \mathbb{R}^d - \mathcal{A}$.

In a similar fashion with the previous arguments, we find

$$\frac{d\eta_m}{dt} \ge -C\eta_m \|\rho\| \left(1 + \eta^{d/p'} \eta_m^{-(1+\alpha)}\right), \ \forall t \in [0, \min(T, T_0^N))$$

Step C: Closing the argument

Until now, we have

$$\begin{cases} \frac{d^{+}\eta}{dt} \leq C\eta \|\rho\| \left(1+\eta^{d/p'}\eta_{m}^{-(1+\alpha)}\right), \\ \frac{d\eta_{m}}{dt} \geq -C\eta_{m} \|\rho\| \left(1+\eta^{d/p'}\eta_{m}^{-(1+\alpha)}\right), \end{cases}$$
(16)

for $t \in [0, \min(T, T_0^N))$. We set

$$f(t) := \frac{\eta(t)}{\eta^0}, \quad g(t) := \frac{\eta_m(t)}{\eta_m^0} \quad \text{and} \quad \xi_N := (\eta^0)^{d/p'} (\eta_m^0)^{-(1+\alpha)}.$$

This yields

$$\frac{d^{+}f}{dt} \leq C \|\rho\| f\left(1 + \xi_{N} f^{d/p'} g^{-(1+\alpha)}\right), \\ \frac{dg}{dt} \geq -C \|\rho\| g\left(1 + \xi_{N} f^{d/p'} g^{-(1+\alpha)}\right).$$

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Since f(0) = g(0) = 1 and $\xi_N \to 0$ as N goes to infinity, we obtain that there exists a positive constant $T^N_* (\leq T^N_0)$ such that

$$\xi_N f^{d/p'} g^{-(1+lpha)} \le 1 \quad {
m for} \quad t \in [0,T^N_*] \,,$$

for sufficiently large N. Then we find

$$f(t) \leq e^{2\|\rho\|t} \quad \text{and} \quad g(t) \geq e^{-2\|\rho\|t}.$$

This yields

$$\xi_N f^{d/p'} g^{-(1+lpha)} \le 1$$
 holds for $t \le -rac{\ln(\xi_N)}{2(d/p' + (1+lpha))\|
ho\|},$

so that

$$-\frac{\ln(\xi_N)}{2(d/p'+(1+\alpha))\|\rho\|} \le T_*^N.$$

On the other hand, our assumption for the initial data (6) implies

$$\liminf_{N \to \infty} T^N_* \ge \lim_{N \to \infty} -\frac{\ln(\xi_N)}{2(d/p' + (1+\alpha)) \|\rho\|} = \infty \,,$$

and thus for N large enough, $T < T^N_\ast < T^N_0.$ This completes the proof.

Local existence and uniqueness of L^p -solutions

Theorem 2. (Local existence and uniqueness of solutions)

Assume that W satisfies the condition (5), for some $0 \le \alpha < \frac{d}{p'} - 1$, and that $\rho^0 \in \mathcal{P}_1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, 1 . Then there exists a time <math>T > 0, depending only on $\|\rho^0\|_p$ and α , and a unique nonnegative solution to (1) satisfying $\rho \in L^{\infty}(0,T; L^1 \cap L^p(\mathbb{R}^d)) \cap \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d))$. Furthermore, the solution satisfies that there exists C > 0 depending only on $\|\rho^0\|_p$ and α such that

$$\|\rho(t)\|_p \le C \qquad \text{for all } t \in [0, T].$$
(17)

The velocity field generated by ρ , given by $u = -\nabla W * \rho$, is bounded and Lipschitz continuous in space uniformly on [0, T], and ρ is determined as the push-forward of the initial density through the flow map generated by u.

Theorem 2. (Continued)

Moreover, if ρ_i , i = 1, 2, are two such solutions to (1) with initial conditions $\rho_i^0 \in \mathcal{P}_1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, 1 , we have the following stability estimate:

 $\frac{d}{dt}d_1(t) \le C \max(\|\rho_1\|, \|\rho_2\|)d_1(t),$

where $d_1(t) := d_1(\rho_1(t), \rho_2(t))$.

Sketch of the proof; Step A.- Uniquness

Given two weak solutions $\rho_i \in L^{\infty}(0,T; L^1 \cap L^p(\mathbb{R}^d)) \cap \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d))$, i = 1, 2, to the continuous aggregation equations (1), consider the two flow maps $\Psi_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, i = 1, 2, generated by the two velocity fields, i.e.,

$$\begin{cases} \frac{d}{dt} \left(\Psi_i(t;s,x) \right) = u_i(t;s,\Psi_i(t;s,x)) \,, \\ \Psi_i(s;s,x) = x, \end{cases}$$

where $u_i := -\nabla W * \rho_i$, $t, s \in [0, T]$ and $x \in \mathbb{R}^d$. We know that the solutions are constructed by transporting the initial measures through the velocity fields $\rho_i = \Psi_i \# \rho_i^0$, i = 1, 2.

Let \mathcal{T}^0 be the optimal transportation between $\rho_1(0)$ and $\rho_2(0)$ for the d_1 -distance. Then we define a transport (not necessarily optimal) between $\rho_1(t)$ and $\rho_2(t)$ by

$$\mathcal{T}^{t}(x) = \Psi_{2}(t; 0, x) \circ \mathcal{T}^{0}(x) \circ \Psi_{1}(0; t, x), \qquad \mathcal{T}^{t} \# \rho_{1}(t) = \rho_{2}(t),$$

and $\frac{d}{dt}d_1(t) \leq Q(t)$, where $d_1(t) := d_1(\rho_1(t), \rho_2(t))$ and

$$Q(t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla W(\mathcal{T}^t(x) - \mathcal{T}^t(y)) - \nabla W(x - y)|\rho_1(t, x)\rho_1(t, y)dxdy ,$$

where we have used a similar argument as in Step A of the proof of Theorem 1. Note by symmetry and the near- and far-field decomposition as in (12) that

$$Q(t) \le 4 \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|\mathcal{T}(x) - x|}{|\mathcal{T}(x) - \mathcal{T}(y)|^{1+\alpha}} + \frac{|\mathcal{T}(x) - x|}{|x - y|^{1+\alpha}} \right) \rho_1(x) \rho_1(y) dx dy$$

$$\le C \max(\|\rho_1\|, \|\rho_2\|) d_1(t).$$

Step B.- Existence

We first regularize ∇W such as $\nabla W_{\varepsilon} := (\nabla W) * \theta_{\varepsilon}$. Then since ∇W_{ε} is a globally Lipschitz, there exists a unique global solution ρ_{ε} to the following system

$$\begin{aligned} \partial_t \rho_{\varepsilon} + \nabla \cdot (\rho_{\varepsilon} u_{\varepsilon}) &= 0, & t > 0, \quad x \in \mathbb{R}^d, \\ u_{\varepsilon}(t, x) &:= -\nabla W_{\varepsilon} * \rho_{\varepsilon}, & t > 0, \quad x \in \mathbb{R}^d, \\ \rho_{\varepsilon}(0, x) &:= \rho^0(x), & x \in \mathbb{R}^d, \end{aligned}$$
(18)

A standard calculation implies that

$$\frac{d}{dt}\|\rho_{\varepsilon}\|_{L^1\cap L^p} \le C\|\rho_{\varepsilon}\|_{L^1\cap L^p}^2,\tag{19}$$

where C is an uniform constant in ε .

Thus we deduce that there exists a T > 0 such that

$$\sup_{\varepsilon>0} \|\rho_{\varepsilon}\| < \infty.$$
⁽²⁰⁾

It follows from (20) and the evolution in time of the first momentum of ρ , that this first moment is also uniformly bounded:

 $\sup_{\varepsilon>0} \|x\rho_{\varepsilon}\|_{L^{\infty}(0,T;L^{1}(\mathbb{R}^{d}))} \leq C,$

where C depends only on $T, ||x\rho^0||_1$, and $||\rho^0||$.

One can use the similar arguments to the previous part to find that

$$\frac{d}{dt}\eta_{\varepsilon,\varepsilon'}(t) \le C \max(\|\rho_{\varepsilon}\|, \|\rho_{\varepsilon'}\|) \left(\eta_{\varepsilon,\varepsilon'}(t) + \varepsilon + \varepsilon'\right),$$
(21)

where C is an uniform constant in ε and ε' . We remark that the above estimate (21) implies that $\{\rho_{\varepsilon}\}_{\varepsilon>0}$ is a Cauchy sequence in $\mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d))$.

 \exists limit curve of measures $\rho \in \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d)) \cap L^{\infty}(0,T; (L^1 \cap L^p)(\mathbb{R}^d))$

Show that ρ satisfies the weak formulation

$\Rightarrow \rho$ is a solution of the aggregation equations (1)

Note that the velocity field is bounded and Lipschitz continuous in space with

 $|u(t,x)-u(t,y)| \le C \|\rho\| |x-y|,$

for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$. Thus, the flow map

$$\left\{ \begin{array}{l} \displaystyle \frac{d}{dt}(\Psi(t;s,x))=u(t;s,\Psi(t;s,x)),\\ \displaystyle \Psi(s;s,x)=x, \end{array} \right.$$

for all $s, t \in [0, T]$ is well-defined.

Choosing as test function in weak formulation $\phi(t,x) = \varphi(\Psi(t;\bar{T},x))$ for any $\bar{T} \in (0,T]$ with $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \rho^0(x) \varphi(\Psi(0;\bar{T},x)) dx = \int_{\mathbb{R}^d} \rho(\bar{T},x) \varphi(x) dx,$$

 $\text{for all } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d) \Rightarrow \rho(\bar{T}) = \Psi(\bar{T}; 0, \cdot \) \# \rho^0 \text{ for any} _{\bar{T}} \in (0, T], \text{ for all } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d) \Rightarrow \rho(\bar{T}) = \Psi(\bar{T}; 0, \cdot \) \# \rho^0 \text{ for any} _{\bar{T}} \in (0, T], \text{ for all } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d) \Rightarrow \rho(\bar{T}) = \Psi(\bar{T}; 0, \cdot \) \# \rho^0 \text{ for any} _{\bar{T}} \in (0, T], \text{ for all } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d) \Rightarrow \rho(\bar{T}) = \Psi(\bar{T}; 0, \cdot \) \# \rho^0 \text{ for any} _{\bar{T}} \in (0, T], \text{ for all } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d) \Rightarrow \rho(\bar{T}) = \Psi(\bar{T}; 0, \cdot \) \# \rho^0 \text{ for any} _{\bar{T}} \in (0, T], \text{ for all } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d) \Rightarrow \rho(\bar{T}) = \Psi(\bar{T}; 0, \cdot \) \# \rho^0 \text{ for any} _{\bar{T}} \in (0, T], \text{ for all } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d) \Rightarrow \rho(\bar{T}) = \Psi(\bar{T}; 0, \cdot \) \# \rho^0 \text{ for any} _{\bar{T}} \in (0, T], \text{ for all } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d) \Rightarrow \rho(\bar{T}) = \Psi(\bar{T}; 0, \cdot \) \# \rho^0 \text{ for any} _{\bar{T}} \in (0, T], \text{ for all } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$

Propagation of chaos

Let us consider $\rho^N(t, x_1, \cdots, x_N)$ being the image by the dynamics to the coupled system (2) with N-equal masses particles of the initial law $(\rho^0)^{\otimes N}$. We define the *k*-marginals as follows.

$$\rho_k^N(t,x_1,\cdots,x_k) := \int_{\mathbb{R}^{d(N-k)}} \rho^N(t,x) dx_{k+1} \cdots dx_N.$$

The propagation of chaos property is defined as follows: for any fixed $k \in \mathbb{N}$,

$$ho_k^N
ightarrow (
ho)^{\otimes k}$$
 weakly-* as measures as $N
ightarrow \infty$.

Theorem 3. (Propagation of chaos)

Given $\rho(t) \in L^{\infty}(0,T;(L^1 \cap L^p)(\mathbb{R}^d)) \cap \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d))$ the unique solution to (1) with initial data $\rho^0 \in \mathcal{P}_1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $1 , <math>d \ge 3$. Assume that ρ^0 has compact support, the initial positions $X^{N,0} := \{X_i^0\}_{i=1}^N$ are iid with law ρ^0 , and

$$(1+\alpha)p' < \frac{p-1}{2p-1}d, \quad \text{with} \quad \alpha \ge 0.$$

Then the propagation of chaos holds in the sense that

$$\mathbb{P}\left(\sup_{t\in[0,T]}d_1(\mu_N(t),\rho(t))\geq \frac{C}{N^{\gamma/d}}\right)\to 0,\quad\text{as}\quad N\to+\infty,$$

where γ is a positive constant satisfying

$$\frac{p'(2p-1)(1+\alpha)}{d(p-1)} < \gamma < 1.$$

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We define the "blob" initial data ρ_N^0 as

$$\rho_N^0 := \mu_N^0 * \frac{\mathbf{1}_{B_{\varepsilon}(0)}}{|B_{\varepsilon}(0)|} = \frac{1}{c_d \varepsilon^d} (\mu_N^0 * \mathbf{1}_{B_{\varepsilon}(0)}),$$
(22)

where $\varepsilon = \varepsilon(N) = N^{-\gamma/d}$, $0 < \gamma < 1$ and c_d is the volume of the unit ball in dimension d. We also define the "blob" approximation $\rho_N(t)$ to be the solution of the system (1) with the initial data ρ_N^0 .

Proposition 1.

Under the assumptions of Theorem 3, and assuming that there exists $C_1 > 0$ independent of the number of particles N such that

$$\|\rho_N^0\|_p \le C_1, \quad \text{and} \quad \eta_m^0 \ge \frac{1}{C_1} \varepsilon^r,$$

with $1 \leq r < \frac{d}{p'(1+\alpha)}$. Then there exists T > 0 such that the solutions $\rho_N(t)$ and the empirical measure $\mu_N(t)$ are well-defined for all $t \in [0, T]$, and

$$d_{\infty}(\rho_N(t),\mu_N(t)) \le d_{\infty}(\rho_N^0,\mu_N^0)e^{C_2T} \le \varepsilon(N)e^{C_2T},$$

where $C_2 > 0$ is independent of N.

Sketch of proof for Theorem 1

We first remark that $\|\rho_N(t)\|_p \leq C$ for all $t \in [0, T]$ where C is independent of N. Then we now use the similar argument as in the proof of Theorem 1 to find

$$\frac{d\eta_N}{dt} \le C\eta_N \left(1 + \eta_N^{d/p'} \eta_m^{-(1+\alpha)} \right),$$

and

$$\frac{d\eta_m}{dt} \ge -C\eta_m \left(1 + \eta_N^{d/p'} \eta_m^{-(1+\alpha)}\right),$$

where $\eta_N(t) := d_{\infty}(\rho_N(t), \mu_N(t))$. Note that the condition $r \ge 1$ makes sense since $\varepsilon \approx \eta_N^0 \ge \eta_m^0 \ge C\varepsilon^r$ for ε small enough. We finally conclude the desired result using the fact that

$$\frac{(\eta^0_N)^{d/p'}}{(\eta^0_m)^{1+\alpha}} \leq C \varepsilon^{d/p'-r(1+\alpha)} \to 0 \quad \text{as} \quad N \to \infty.$$

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Lemma 1.

Let $\rho^0 \in \mathbb{P}_1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), 1 , and the initial positions <math>X^{N,0}$ be iid with law ρ^0 . Suppose that there exists L > 0 such that

$$2c_d^{\frac{1}{p'}} \|\rho^0\|_p L^d p' \le N,$$

then η_m^0 satisfies

$$\mathbb{P}\left(\eta_{m}^{0} \geq LN^{-\frac{2p-1}{d(p-1)}}\right) \geq e^{-2c_{d}^{\frac{1}{p'}} \|\rho^{0}\|_{p}L^{\frac{d}{p'}}}$$

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Lemma 2.

Let $\rho^0 \in \mathbb{P}_1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), 1 with compactly support included in <math>[-R, R]^d$. For any iid $X^{N,0}$ with law ρ^0 , the smoothed empirical measures ρ^0_N defined in (22) satisfy the explicit *large deviations* bound

$$\mathbb{P}(L_d \| \rho^0 \|_p \le \| \rho_N^0 \|_p) \le [2(R+1)]^d N^{\gamma} e^{-c_R \| \rho \|_p N^{1-\gamma}},$$

where L_d and c_R are explicitly given by

$$c_R := \frac{2 \ln 2}{\left[2(R+1)\right]^{\frac{d}{p}}} \quad \text{and} \quad L_d := \frac{4(4[[\sqrt{d}]]+1)^{d/p}}{c_d},$$

where $[[\cdot]]$ denoting the integer part.

Sketch of proof for Theorem 3

We introduce several sets for the random initial data:

$$w_1 := \{ X^{N,0} : \eta_m^0 \ge \varepsilon^r \}, \quad w_2 := \{ X^{N,0} : L_d \| \rho^0 \|_p \ge \| \rho_N^0 \|_p \},$$

and

$$w_3 := \{ X^{N,0} : d_1(\mu_N^0, \rho^0) \le \varepsilon \},\$$

where r, ε , and L_d are given in the previous estimates. We can find that²

$$\mathbb{P}(w_1^c) \leq C N^{-\frac{d\beta}{p'}}, \quad \mathbb{P}(w_2^c) \leq C N^{\gamma} e^{-C N^{1-\gamma}}, \quad \text{and} \quad \mathbb{P}(w_3^c) \leq C N^{-s'},$$

where C, β , and s' are positive constants. We now denote $w := w_1 \cap w_2 \cap w_3$. Then we have

$$\mathbb{P}(w^c) \le CN^{-l}, \quad \text{for some} \quad C, l > 0.$$

²Boissard(2011)

If the initial data belongs to w, then we obtain from Proposition 1 that

$$d_1(\rho_N(t), \mu_N(t)) \le d_{\infty}(\rho_N(t), \mu_N(t)) \le \frac{Ce^{CT}}{N^{\gamma/d}}, \text{ for } t \in [0, T].$$

We can also notice from Theorem 1 that

$$d_1(
ho(t),
ho_N(t)) \leq rac{Ce^{CT}}{N^{\gamma/d}} \quad ext{for all} \quad t\in[0,T].$$

Hence we have

$$\mathbb{P}(w) \le \mathbb{P}\left(\sup_{t \in [0,T]} d_1(\rho(t), \mu_N(t)) \le \frac{Ce^{CT}}{N^{\gamma/d}}\right),\,$$

and it implies that

$$\mathbb{P}\left(\sup_{t\in[0,T]}d_1(\rho(t),\mu_N(t))\geq \frac{Ce^{CT}}{N^{\gamma/d}}\right)\leq \mathbb{P}(w^c)\leq \frac{C}{N^l}.$$

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This completes the proof.

Thank you for your attention

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