Of Chaos and Chaotic States in Kac's Model.

Amit Einav, University of Cambridge

Microscopic Descriptions and Mean Field Equations in Physics and Social Sciences, University of Bath

15th of May, 2014

Joint Work with Kleber Carrapatoso

Kac's Model

- In 1956 Marc Kac introduced an $N-$particle linear model from which a caricature of the Boltzmann equation arose as a mean field limit.
- Kac's model consisted of N indistinguishable particles with one dimensional velocity that undergo random collision in the following manner: Suppose that a moment before the collision the velocity vector of the ensemble was $(v_1, \dots, v_N)$. When the collision occurs, we pick a pair of particles at random, say (v_i, v_j), with equal probability for any i, j, and collide them. After their collision the new velocity vector is given by $(v_1, \dots, v_i(\vartheta), \dots, v_j(\vartheta), \dots, v_N)$ where

$$\begin{pmatrix} v_i(\vartheta) \\ v_j(\vartheta) \end{pmatrix} = \begin{pmatrix} v_i \cos\vartheta + v_j \sin\vartheta \\ -v_i \sin\vartheta + v_j \cos\vartheta \end{pmatrix} = R_\vartheta \begin{pmatrix} v_i(\vartheta) \\ v_j(\vartheta) \end{pmatrix}$$

Kac's Master Equation.

- Denoting by

$$Q(\psi)(\mathbf{v}) = \frac{1}{\binom{N}{2}} \sum_{i<j} \frac{1}{2\pi} \int_0^{2\pi} \psi\left(R_{i,j,\vartheta}\mathbf{v}\right) d\vartheta,$$

one finds that the evolution equation for the $N-$particles distribution function, F_N, is given by

$$\frac{\partial F_N}{\partial t}\left(v_1, \dots, v_N, t\right) = N(Q-I)F_N\left(v_1, \dots, v_N, t\right), \tag{1}$$

This equation is usually called Kac's Master equation.

- Kac's Master equation can be considered on $\mathbb{R}^N$, but a more realistic approach would be to restrict it to the manifold $\mathbb{S}^{N-1}(\sqrt{N})$ (which we will call 'Kac's sphere') where Q is a bounded, self adjoint operator satisfying $Q \leq I$ and

$$\mathrm{Ker}\left(I-Q\right) = \mathrm{span}\left\{1\right\}.$$

Connection to Boltzmann's Equation and Chaoticity.

- Integrating over all the velocities but v_1 in Kac's Master equation, and using symmetry one can see that

$$\partial_t \Pi_1(F_N)(v_1) = \frac{1}{\pi}\int_0^{2\pi}\int_{\mathbb{R}} \left(\Pi_2(F_N)(v_1(\vartheta), v_2(\vartheta)) - \Pi_2(F_N)(v_1, v_2)\right) d\vartheta dv_2$$

  where $\Pi_k(F_N)$ is the $k-$th marginal.
- The above equation looks very similar to a Boltzmann type equation if $\Pi_2(F_N) \approx \Pi_1(F_N) \otimes \Pi_1(F_N)$.

Definition

Let X be a Polish space. A family of symmetric probability measures on X^N, $\{\mu_N\}_{N\in\mathbb{N}}$, is called $\mu-$chaotic, where μ is a probability measure on X, if for any $k \in \mathbb{N}$

$$\lim_{N\to\infty} \Pi_k(\mu_N) = \mu^{\otimes k},$$

where $\Pi_k(\mu_N)$ is the $k-$th marginal of μ_N, and the limit is taken in the weak topology.

- Under the assumption of $f-$chaoticity on the family of density functions $\{F_N\}_{N\in\mathbb{N}}$ (i.e. assuming that the family $d\mu_N = F_N d\sigma^N$ is $d\mu = f(v)dv-$chaotic, where σ^N is the uniform probability measure on Kac's sphere), our marginal equation turns into the Boltzmann-Kac equation:

$$\partial_t f(v_1) = \frac{1}{\pi}\int_0^{2\pi}\int_{\mathbb{R}} \left(f(v_1(\vartheta))f(v_2(\vartheta)) - f(v_1)f(v_2)\right) d\vartheta dv_2.$$

- Kac showed that if the initial data is chaotic, then the solution to his Master equation is also chaotic for all $t > 0$. Moreover, the limit function $f(t,v)$ satisfies the above caricature of the Boltzmann equation.

The Existence of Chaotic States.

- There is a simple formula to create chaotic states on Kac's sphere. Given a 'nice' enough distribution function, $f(v)$, on $\mathbb{R}$, one defines the following distribution function on Kac's sphere:

$$F_N(v_1,\dots,v_N) = \frac{\Pi_{i=1}^N f(v_i)}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}, \tag{2}$$

where $\mathcal{Z}_N\left(f,\sqrt{N}\right) = \int_{\mathbb{S}^{N-1}(\sqrt{N})} \Pi_{i=1}^N f(v_i) d\sigma^N$. We call distributions of the form (2) *conditioned tensorisation of* f.

- Kac showed that under very restrictive integrability conditions on f, the above family is indeed $f-$chaotic. This was extended in 2010 by Carlen, Carvalho, Le Roux, Loss and Villani who managed to show the following:

Theorem

Let f be a distribution function on $\mathbb{R}$ such that $f \in L^p(\mathbb{R})$ for some $p>1$, $\int_{\mathbb{R}} v^2 f(v)dv = 1$ and $\int_{\mathbb{R}} v^4 f(v)dv < \infty$. Then the family of conditioned tensorisation of f is $f-$chaotic.

The Normalisation Function.

- The proof of Carlen et al's theorem relies *heavily* on an asymptotic approximation of the normalisation function

$$\mathcal{Z}_N\left(f,\sqrt{r}\right)=\int_{\mathbb{S}^{N-1}(r)}\Pi_{i=1}^N f(v_i)d\sigma_r^N,$$

where $d\sigma_r^N$ is the uniform probability measure on the sphere $\mathbb{S}^{N-1}(r)$, as it measures how well the tensoriastion of f is concentrated on the sphere of radius r.

Theorem (Carlen et. al. 2010)

Let f be a distribution function on $\mathbb{R}$ such that $f\in L^p(\mathbb{R})$ for some $p>1$, $\int_{\mathbb{R}} v^2 f(v)dv=1$ and $\int_{\mathbb{R}} v^4 f(v)dv<\infty$. Then

$$\mathcal{Z}_N(f,\sqrt{r})=\frac{2}{\sqrt{N}\Sigma\left|\mathbb{S}^{N-1}\right|r^{\frac{N-2}{2}}}\left(\frac{e^{-\frac{(r-N)^2}{2N\Sigma^2}}}{\sqrt{2\pi}}+\lambda_N(r)\right),$$

where $\Sigma^2=\int_{\mathbb{R}} v^4 f(v)dv-1$ and $\sup_r\left|\lambda_N(r)\right|\underset{N\to\infty}{\longrightarrow}0$.

Normalisation Function Cont.

- This shows, as one can expect from the Law of Large Numbers, a concentration of $f^{\otimes N}$ on Kac's sphere when the second moment of f is 1. However, Carlen et. al. managed to give a quantitative concentration result which corresponds to a local central limit theorem. Indeed, the normalisation function can be written as

$$\mathcal{Z}_N(f, \sqrt{r}) = \frac{2h^{*N}(r)}{\left|\mathbb{S}^{N-1}\right| r^{\frac{N-2}{2}}},$$

where h is the distribution function of the random variable V^2, obtained from f via a simple transformation.

- As the fourth moment of f equals the second moment of h, it is no surprise that it is a part of the requirement for a *Gaussian* local central limit theorem. However, in showing chaoticity, and a stronger sense of chaoticity we'll discuss later, what matters is *not* the exponential concentration about Kac's sphere - but the fact that there is concentration about Kac's sphere. This leads to the current investigation of conditioned tensorisation of f, where f has moments of order 2α with $1 < \alpha < 2$.

Lévy Type Local Central Limit Theorem.

- The Gaussian is not the only 'attractor' in the sense of central central limit theorems. A different type of attractor, called $\alpha-$stable distribution, is a random variable X whose characteristic functions behaves like

$$\varphi_X(\xi) = e^{-|\xi|^{\alpha}},$$

for $0 < \alpha < 2$. This ensures the self similarity condition

$$\varphi(\xi) = \left(\varphi\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)\right)^N,$$

corresponding the an appropriate rescaling on the $N-$times convolution of X.

- In general, we say that a random variable X is an $\alpha-$stable distribution for $0 < \alpha < 2$, $\alpha \neq 1$, if there exists $\sigma > 0$ and $\beta \in [-1, 1]$ such that the characteristic function of X satisfies

$$\widehat{\gamma}_{\sigma,\alpha,\beta}(\xi) = e^{-\sigma|\xi|^{\alpha}\left(1+i\beta \text{sgn}(\xi)\tan\frac{\alpha\pi}{2}\right)}.$$

Lévy Local Central Limit Theorem Cont.

Theorem (Carrapatoso and E. 2013)

Let g be the probability density function of a random real variable X with zero mean. Assume that $g \in L^p(\mathbb{R})$ for some $p > 1$, and that there exists $C > 0$ such that

$$\mu_g(x) = \int_{-x}^{x} y^2 g(y)dy \underset{x\to\infty}{\approx} Cx^{2-\alpha}$$

for some $1 < \alpha < 2$. Assume in addition that

$$\frac{1 - G(x)}{1 - G(x) + G(-x)} \underset{x\to\infty}{\longrightarrow} p, \quad \frac{G(-x)}{1 - G(x) + G(-x)} \underset{x\to\infty}{\longrightarrow} q$$

where $G(x) = \int_{-\infty}^{x} g(y)dy$, and that g has finite moment of some order. Define

$$g_N(x) = N^{\frac{1}{\alpha}} g^{*N}\left(N^{\frac{1}{\alpha}} x\right),$$

and

$$\gamma_{\sigma,\alpha,\beta}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\gamma}_{\sigma,\alpha,\beta}(\xi) e^{i\xi x} d\xi,$$

with $\sigma = C\frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}\cos\left(\frac{(\alpha)}{2}\right)$ and $\beta = p - q$.

Lévy Local Central Limit Theorem Cont.

Then, for any positive sequence $\{\beta_N\}_{N\to\infty}$ that converges to zero as N goes to infinity, any $\tau > 0$ and N large enough we have that

$$\left\|g_N - \gamma_{\sigma,\alpha,\beta}\right\|_\infty \leq C_{g,\alpha}\Bigg(N^{\frac{1}{\alpha}}\left(1-\beta_N^{2+\tau}+\phi_\tau(\beta_N)\right)^{N-q} + e^{-\frac{\sigma N \beta_N^\alpha}{2}}$$

$$+\omega_g(\beta_N) + 2\sigma\beta_N^\alpha\left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)\Bigg) = \epsilon_\tau(N),$$

where

(i) $C_{g,\alpha} > 0$ is a constant depending only on g, its moments and α.

(ii) q can be chosen to be the Hölder conjugate of $\min(2,p)$.

(iii) ϕ_τ satisfies

$$\lim_{x\to 0}\frac{\phi_\tau(x)}{|x|^{2+\tau}} = 0,$$

(iv) $\omega_g(\beta) = \sup_{|\xi|<\beta}\frac{\left|\widehat{g}(\xi)-1-\sigma|\xi|^\alpha\left(1+i\beta\mathrm{sgn}(\xi)\tan\frac{\alpha\pi}{2}\right)\right|}{|\xi|^\alpha}$ satisfies

$$\lim_{\beta\to 0}\omega_g(\beta) = 0.$$

Normalisation Function Estimation

Theorem (Carrapatoso and E. 2013.)

Let f be the probability density function of a random real variable V such that $f \in L^p(\mathbb{R})$ for some $p > 1$ and let

$$\nu_f(x) = \int_{-\sqrt{x}}^{\sqrt{x}} y^4 f(y)dy.$$

Assume that

$$\int_{\mathbb{R}} x^2 f(x)dx = E < \infty.$$

and $\nu_f(x) \underset{x\to\infty}{\sim} Cx^{2-\alpha}$ for some $C > 0$ and $1 < \alpha < 2$. Then

$$\mathcal{Z}_N\left(f, \sqrt{r}\right) = \frac{2}{\left|\mathbb{S}^{N-1}\right| r^{\frac{N-2}{2}}} \frac{1}{N^{\frac{1}{\alpha}}} \left(\gamma_{\sigma,\alpha,1}\left(\frac{r - NE}{N^{\frac{1}{\alpha}}}\right) + \lambda_N(r)\right),$$

where $\sup_u \left|\lambda_N(u)\right| \underset{N\to\infty}{\longrightarrow} 0$.

Entropic Chaos.

- In their work, Carlen et al have defined a new, more robust, concept of chaoticty - *entropic chaoticity*. Motivated by the attempt to investigate the rate of convergence to equilibrium in the Boltzmann-Kac equation via Kac's many particle model, it was noticed that the inherent L^2 norm and associated spectral gap problem is not adequate to deal with chaotic states. A different type of 'distance' was needed to investigate the trend to equilibrium. Taking a leaf form Boltzmann's research one can define the entropy on Kac's sphere as

$$H_N(F_N) = \int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N \log F_N d\sigma^N.$$

For 'natural' $f-$chaotic states, one can imagine that $F_N \approx f^{\otimes N}$ in some sense. Plugging this into the entropy yields

$$H_N(F_N) \approx N \int_{\mathbb{R}} f(v) \log\left(\frac{f(v)}{\gamma(v)}\right) dv = NH(f|\gamma),$$

where γ is the normal Gaussian. We see that in the above expression *all* the particles (and correlations) come into play, which is much stronger than the normal chaoticity definition.

Entropic Chaos Cont.

- This leads us to define the following

Definition

A family of densities $\{F_N\}_{N\in\mathbb{N}}$ on Kac's sphere is called $f-$entropically chaotic if it is $f-$chaotic and

$$\lim_{N\to\infty} \frac{H_N(F_N)}{N} = H(f|\gamma).$$

- The concept of entropic chaoticity plays an important role in recent studies of trend to equilibrium. Moreover, conditioned tensorisation of a function f, for f that satisfies our theorem's condition, are $f-$entropically chaotic and play a sort of 'attractors' in the setting of entropically chaotic families in the following sense: If μ_N is a family of symmetric measures on Kac's sphere such that

$$\lim_{N\to\infty} \frac{H_N\left(\mu_N|F_N\right)}{N} = 0,$$

where F_N is the conditioned tensorisation of f, then μ_N is $f-$entropically chaotic.

Idea of the Proof.

- One can show that the $k-$th marginal of F_N is given by the density

$$\Pi_k\left(F_N\right)\left(v_1,\dots,v_k\right)=\frac{\left|\mathbb{S}^{N-k-1}\right|}{\left|\mathbb{S}^{N-1}\right|}\frac{\left(N-\sum_{i=1}^k v_i^2\right)^{\frac{N-k-2}{2}}}{N^{\frac{N-2}{2}}}$$

$$\frac{\mathcal{Z}_{N-k}\left(f,\sqrt{N-\sum_{i=1}^k v_i^2}\right)}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}f^{\otimes k}\left(v_1,\dots,v_k\right)$$

- Using our approximation theory one gets

$$\Pi_k\left(F_N\right)\left(v_1,\dots,v_k\right)=\left(\frac{N}{N-k}\right)^{\frac{1}{\alpha}}$$

$$\frac{\gamma_{\sigma,\alpha,1}\left(\frac{k-\sum_{i=1}^k v_i^2}{(N-k)^{\frac{1}{\alpha}}}\right)+\lambda_{N-k}\left(N-\sum_{i=1}^k v_i^2\right)}{\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)}f^{\otimes k}\left(v_1,\dots,v_k\right)$$

Idea of the Proof Cont.

- As $\gamma_{\sigma,\alpha,1}$ is bounded, we can use the dominated convergence theorem to show that
$$\Pi_k\left(F_N\right) \underset{N\to\infty}{\longrightarrow} f^{\otimes k}$$
in $L^1\left(\mathbb{R}^k\right)$, which is stronger than in the weak topology. This shows chaoticity.
- For entropic chaoticity one can show that

$$\frac{H_N(F_N)}{N} = \left(\frac{N}{N-1}\right)^{\frac{1}{\alpha}} \int_{-\sqrt{N}}^{\sqrt{N}} \frac{\gamma_{\sigma,\alpha,1}\left(\frac{1-v^2}{(N-1)^{\frac{1}{\alpha}}}\right) + \lambda_{N-1}\left(N - v^2\right)}{\gamma_{\sigma,\alpha,1}(0) + \lambda_N(N)}$$
$$f(v)\log f(v)dv - \frac{\log\left(\frac{2\left(\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)\right)}{\left|\mathbb{S}^{N-1}\right| N^{\frac{N-2}{2}}} N^{\frac{1}{\alpha}}\right)}{N}$$

and taking N to infinity, one gets $H(f|\gamma)$, as desired.

Final Remarks.

- Much of the current research around Kac's model surrounds the attempt to show an exponential rate of decay to equilibrium that is uniform in N (the so called Cercignani many body conjecture). That way, one can hope to pass to the limit and get the same decay rate for the Boltzmann-Kac equation. Conditioned tensorition played a role in showing that in general this isn't true. However, there is a tremulous hope in the community that some restrictions on the underlying distribution function may lead to a positive result. This problems is of great interest to us.
- While it seems that the entropy is a good functional to investigate trend to equilibrium, there are other extensive functional on Kac's sphere that may prove to be better tools to measure convergence to equilibrium. One prominent example is the Wasserstein distance, which seems to appear a lot in Kinetic Theory and has been used extensively in a recent remarkable work by Mouhot and Mischler to solve many unkown problems with regards to propagation of chaos and convergence to equilibrium.

References.

- Carlen E. A., Carvalho M. C., Le Roux J., Loss M. and Villani C.: Entropy and Chaos in the Kac Model. *Kinet. Relat. Models*, **3** (2010), no. 1, 85–122.
- Carrapatoso K.: Quantitative and Qualitative Kac's Chaos on the Boltzmann Sphere. http://arxiv.org/abs/1205.1241.
- Einav A.: On Villani's Conjecture Concerning Entropy Production for the Kac Master Equation. *Kinet. Relat. Models*, **4** (2011), no. 2, 479–497.
- Goudon T., Junca S. and Toscani G.: Fourier Based Distances and Berry-Esseen Like Inequalities for Smooth Densities. *Monatsh. Math.*, **135** (2002), no. 2, 115–136.
- Hauray M. and Mischler S.: On Kac's Chaos and Related Problems. http://hal.archives-ouvertes.fr/hal-00682782/.
- Mischler S. and Mouhot C.: Kac's Program in Kinetic Theory. *Inventiones Mathematicae*, **193** (2013), issue 1, pp 1–147.

Thank You!