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Boundary conditions for measure-valued evolutions

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We aim to derive suitable boundary conditions for the evolution of measures on bounded domains. We restrict ourselves to a *simplified* scenario in which mass moves on an interval $[0, 1]$, stops at $x = 1$ and disappears from there at a certain rate. The object of interest is the mass measure μ_t on $[0, 1]$ that evolves in time.

Flow

The velocity $v : [0, 1] \to \mathbb{R}$ is prescribed and is only space-dependent, as is illustrated by Figure 1, in which a point mass (ball) starts at x_0 and follows a characteristic. Once it hits the right-hand boundary $x = 1$ at time $t = \tau_\partial(x_0)$, it sticks there.

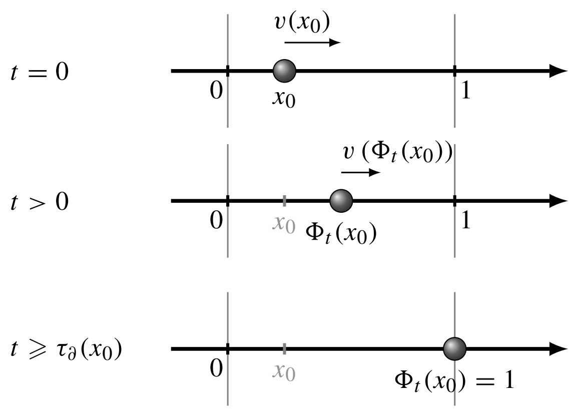

Figure 1: 'Individualistic flow': a point mass moves along a characteristic.

This process is described by the semigroup Φ_t, such that $\Phi_t(x_0)$ is the position at time t of a point mass that started at position x_0. The semigroup P_t is the lift of Φ_t to the space of Borel measures by means of a push-forward:

$$\mu_t = P_t\, \mu_0 := \Phi_t \# \mu_0 = \mu_0 \circ \Phi_t^{-1}.$$

Here, μ_0 denotes the initial mass measure.

Absorption

At $x = 1$ mass accumulates and is then taken away at rate a. We introduce a regularization: absorption of mass takes place in a boundary *layer* of width $1/n$ around $x = 1$. The corresponding solution is denoted by $\mu_t^{(n)}$. The function f_n in Figure 2 is used to incorporate absorption of mass on a layer. The total 'sink of mass' is given by

$$F^{(n)}\mu := -a\, f_n\, \mu.$$

This term can formally be interpreted as the right-hand side of a continuity equation that describes the evolution of $\mu_t^{(n)}$.

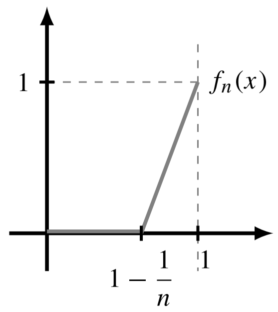

Figure 2: Function f_n used for regularization.

Equations

We look for a *mild solution* of the evolution of $\mu_t^{(n)}$ under the given flow and absorption in the boundary layer. Such solution satisfies (by definition) the variation of constants formula

$$\mu_t^{(n)} = P_t\, \mu_0 + \int_0^t P_{t-s}\, F^{(n)} \mu_s^{(n)}\, ds.$$

Formally, as $n \to \infty$ we obtain

$$\mu_t = P_t\, \mu_0 - a \int_0^t \mu_s(\{1\})\, ds \cdot \delta_1,$$

or, in shorthand notation,

$$\frac{\partial}{\partial t}\mu_t + \frac{\partial}{\partial x}(v\, \mu_t) = -a\, \mu_t(\{1\})\, \delta_1.$$

We made this passage to the limit rigorous.

Results

The main results of [1, 2] are:

- Existence and uniqueness for the regularized problem;
- Existence and uniqueness for the limit problem;
- Convergence rate: $||\mu_t^{(n)} - \mu_t|| = \mathcal{O}(\frac{1}{n})$ in the dual bounded Lipschitz norm;
- Continuous dependence on initial conditions.

References

[1] J.H.M. Evers, S.C. Hille and A. Muntean. "Well-posedness and approximation of a measure-valued mass evolution problem with flux boundary conditions". *Comptes Rendus Mathématique*, 352, 51-54 (2014).

[2] J.H.M. Evers, S.C. Hille and A. Muntean. "Solutions to a measure-valued mass evolution problem with flux boundary conditions inspired by crowd dynamics". `arXiv:1210.4118`.