NONLOCAL INTERACTION EQUATIONS WITH TWO SPECIES

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ABSTRACT

We presents a systematic existence and uniqueness theory of weak measure solutions for systems of nonlocal interaction PDEs with two species, which are the PDE counterpart of systems of deterministic interacting particles with two species. The main motivations behind those models arise in cell biology, pedestrian movements and opinion formation. In case of symmetrizable systems (i. e. with cross-interaction potentials one multiple of the other), we provide in [3] a complete existence and uniqueness theory within (a suitable generalization of) the Wasserstein gradient flow theory in [1], which allows to consider interaction potentials with discontinuous gradient at the origin, see [2]. In the general case of non symmetrizable systems, we provide in [3] an existence result for measure solutions which uses a implicit-explicit version of the JKO scheme, which holds in a reasonable non-smooth setting for the interaction potentials. Uniqueness in the non symmetrizable case is proven for smooth potentials using a variant of the method of characteristics. One-dimensional local nonlinear stability for a nonlocal predator-prey model is discussed in [4], both at particles and PDE levels providing some numerical results.

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INTRODUCTION

Several phenomena in particle physics, cell and population biology, and social sciences, can be modelled by a discrete set of N *interacting agents*, or *particles*. We focus on models with *more than one species*. Assume $X_1,\dots,X_N$ are particles of the first species and $Y_1,\dots,Y_M$ are particles of the second species, the movement of the particles can be described through the Cauchy problem on

$$\begin{cases}\dot X_i(t) = -\sum_{k\neq i} n_k\nabla H_1(X_i(t)-X_k(t)) - \sum_k m_k\nabla K_1(X_i(t)-Y_k(t))\\ \dot Y_j(t) = -\sum_{k\neq j} m_k\nabla H_2(Y_j(t)-Y_k(t)) - \sum_k n_k\nabla K_2(Y_j(t)-X_k(t))\end{cases} \tag{1}$$

with $i=1,\dots,N$ and $j=1,\dots,M$. Denoting with $\mu_1,\mu_2\in\mathcal{P}(\mathbb{R}^d)$ the empirical measures of the sets X_j's and Y_j's respectively, one easily obtain the following system as continuum PDE counterpart of (1)

$$\begin{cases}\partial_t\mu_1 = \mathrm{div}\,(\mu_1\nabla H_1*\mu_1 + \mu_1\nabla K_1*\mu_2)\\ \partial_t\mu_2 = \mathrm{div}\,(\mu_2\nabla H_2*\mu_2 + \mu_2\nabla K_2*\mu_1)\,.\end{cases} \tag{2}$$

In (1) and (2), H_1 and H_2 are called *self-interaction* potentials, whereas K_1 and K_2 are called *cross-interaction* potentials.

WASSERSTEIN DISTANCE:

We denote with $\mathcal{P}(\mathbb{R}^d)$ the space of all the probability measures on $\mathbb{R}^d$ and with $\mathcal{P}_2(\mathbb{R}^d)$ the space of probability measures with finite second moment. We endow the space $\mathcal{P}_2(\mathbb{R}^d)$ with the Wasserstein distance, cf. for instance [1]

$$W_2^2(\mu,\nu) = \int_{\mathbb{R}^{2d}}|x-y|^2 d\gamma(x,y), \qquad \gamma\in\Gamma_o(\mu,\nu).$$

where $\Gamma_o(\mu,\nu)$ as the class of optimal plans, between μ and ν. In order to match the 'multi-species' structure (2) of our modeling setting, we shall work in the product space $\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)$. We shall use bold symbols to denote elements in a product space. For instance, we use $\boldsymbol{\mu}=(\mu_1,\mu_2)\in\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)$, $\boldsymbol{x}=(x_1,x_2)\in\mathbb{R}^d\times\mathbb{R}^d$. Let $\alpha>0$ be fixed. For all $\boldsymbol{\mu},\boldsymbol{\nu}\in\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)$, we define the *$\alpha$-product Wasserstein distance* as follows

$$\mathcal{W}_{2,\alpha}^2(\boldsymbol{\mu},\boldsymbol{\nu}) = W_2^2(\mu_1,\nu_1) + \frac{1}{\alpha}W_2^2(\mu_2,\nu_2).$$

SYMMETRIZABLE SYSTEMS AND GRADIENT FLOW STRUCTURE

Our first aim is to address the case in which there exists a constant $\alpha>0$ such that

$$K_2 = \alpha K_1. \tag{3}$$

We shall call *symmetrizable systems* those which satisfy condition (3), that can be cast in a variational Wasserstein gradient flow approach by means of the *interaction energy* functional

$$\mathcal{F}(\mu_1,\mu_2) = \frac{1}{2}\int_{\mathbb{R}^d} H_1*\mu_1 d\mu_1 + \frac{1}{2}\int_{\mathbb{R}^d} H_2*\mu_2 d\mu_2 + \int_{\mathbb{R}^d} K*\mu_2 d\mu_1. \tag{4}$$

More precisely, the system (2) can be formally written as

$$\begin{cases}\partial_t\mu_1 = \mathrm{div}\left(\mu_1\nabla\dfrac{\delta\mathcal{F}}{\delta\mu_1}\right)\\ \partial_t\mu_2 = \alpha\,\mathrm{div}\left(\mu_2\nabla\dfrac{\delta\mathcal{F}}{\delta\mu_2}\right).\end{cases} \tag{5}$$

MAIN ASSUMPTIONS:

A function $K:\mathbb{R}^d\to\mathbb{R}$ is called an *admissible potential* if $K\in C(\mathbb{R}^d)$ and $K(0)=0$ and $K(-x)=K(x)$. An admissible potential K is said to be *λ-convex* for some $\lambda\in\mathbb{R}$ if the map $\mathbb{R}^d\ni x\mapsto K(x)-\frac{\lambda}{2}|x|^2\in\mathbb{R}$ is convex. K is said to be *mildly singular* if

(MS) $K\in C^1(\mathbb{R}^d\setminus\{0\})$.

K is said to be *sub-quadratic at infinity* if there exists a constant $C>0$ such that

(SQ) $K(x)\le C(1+|x|^2)$ for all $x\in\mathbb{R}^d$.

K is said to be an *attractive non-Osgood potential* if it is radial, i.e. there exists a function k such that $K(x)=k(|x|)$, k is increasing on $r>0$, and the function $[0,+\infty)\ni r\mapsto k'(r)/r$ is non increasing and the non-Osgood condition holds for some $\epsilon>0$:

$$\int_0^\epsilon\frac{dr}{k'(r)}<\infty. \tag{6}$$

GRADIENT FLOW THEORY AND FINITE TIME BLOW UP:

Definition 1. *We say that an absolutely continuous curve $\boldsymbol{\mu}_t = (\mu_{1,t},\mu_{2,t}):[0,T]\to\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)$ is a gradient flow for $\mathcal{F}$ if $\mu_{1,t}$ and $\mu_{2,t}$ solve the system of two continuity equations:*

$$\begin{aligned}\partial_t\mu_{1,t} &= \mathit{div}\,(\mu_1 v_{1,t})\\ \partial_t\mu_{2,t} &= \mathit{div}\,(\mu_2 v_{2,t})\,,\end{aligned} \tag{7}$$

with $v_{i,t}$ in the sub-differentials of $\mathcal{F}$.

We obtain the existence of solutions by means of the Jordan-Kinderlehrer-Otto (JKO) scheme: given an initial product measure $\boldsymbol{\mu}_0\in\mathcal{P}(\mathbb{R}^d)\times\mathcal{P}(\mathbb{R}^d)$ and a time step $\tau>0$, we define the recursive sequence $\boldsymbol{\mu}_\tau^k$ via $\boldsymbol{\mu}_\tau^0=\boldsymbol{\mu}_0$ and

$$\boldsymbol{\mu}_\tau^{k+1}\in\mathrm{argmin}_{\boldsymbol{\mu}\in\mathcal{P}_2(\mathbb{R}^d)^2}\left\{\frac{1}{2\tau}\mathcal{W}_2^2(\boldsymbol{\mu}_\tau^k,\boldsymbol{\mu})+\mathcal{F}[\boldsymbol{\mu}]\right\}.$$

Theorem 1. *Let K_{ij} be admissible λ_{ij} convex potentials satisfying (SQ) and (MS), and let $\boldsymbol{\mu}_t$ be a gradient flow solution to* (2) *according to Definition 1. Then, $\boldsymbol{\mu}_t$ satisfies the following* Evolution Variational Inequality (E. V. I.)

$$\frac{1}{2}\frac{d}{dt}\mathcal{W}_2^2(\boldsymbol{\mu}_t,\boldsymbol{\nu}) + \frac{\lambda}{2}\mathcal{W}_2^2(\boldsymbol{\mu},\boldsymbol{\nu}) \le \mathcal{F}(\boldsymbol{\nu}) - \mathcal{F}(\boldsymbol{\mu}), \tag{8}$$

for all $\boldsymbol{\nu}\in\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{P}_2(\mathbb{R}^d)$. In addition, given two gradient flow solutions $\boldsymbol{\mu}^1$, $\boldsymbol{\mu}^2$ corresponding to the initial data $\boldsymbol{\mu}_0^1$ and $\boldsymbol{\mu}_0^2$, we have the $|\lambda|$-contraction property in $\mathcal{W}_2$

$$\mathcal{W}_2^2(\boldsymbol{\mu}_t^1,\boldsymbol{\mu}_t^2)\le e^{|\lambda|t}\mathcal{W}_2^2(\boldsymbol{\mu}_0^1,\boldsymbol{\mu}_0^2). \tag{9}$$

Theorem 2. *Let K_{ij} be admissible, λ_{ij} convex potentials satisfying (SQ), (MS), and let $\boldsymbol{\mu}(t)$ the unique gradient flow solution to* (2) *with initial datum $\boldsymbol{\mu}_0=(\mu_{0,1},\mu_{0,2})$.*

1. *Assume that all the potentials K_{ij} are attractive non-Osgood potential, and assume $\boldsymbol{\mu}_0$ is supported in $\bar B(X_C,R_0)\times\bar B(Y_C,R_0)$. Then, there exists T^* depending only on R_0 such that $\boldsymbol{\mu}(t)=(\delta_{C_M},\delta_{C_M})\quad\forall t\ge T^*$.*
2. *Assume that K_{11} and K_{22} are attractive non-Osgood potential and K_{12} is admissible, radial and satisfies k_{12}' non-increasing, and*

$$|\nabla K_{12}(x)|\to 0 \text{ for } |x|\to+\infty. \tag{10}$$

. Then that there exist $0<T^*<\bar T$ and $0<R_1<R_2$ such that, if:

- *$\mu_{0,1},\mu_{0,2}$ are supported in $\bar B(x_C,R_1)$ and $\bar B(y_C,R_1)$ respectively,*
- *$d\left(\bar B(x_C,R_1),\bar B(y_C,R_0)\right)\ge R_2$;*

then

$$\boldsymbol{\mu}(t) = (\delta_{x_C},\delta_{y_C})\quad\forall t\in[T^*,\bar T],$$

and $\boldsymbol{\mu}(t)=(\delta_{C_M},\delta_{C_M})\quad\forall t\ge\bar T$.

NON-SYMMETRIZABLE SYSTEMS: WELL-POSEDNESS VIA IMPLICT-EXPLICIT EULER SCHEME

Let us consider the general system

$$\begin{cases}\partial_t\mu_1 = \mathrm{div}\,(\mu_1\nabla H_1*\mu_1 + \mu_1\nabla K_1*\mu_2)\\ \partial_t\mu_2 = \mathrm{div}\,(\mu_2\nabla H_2*\mu_2 + \mu_2\nabla K_2*\mu_1)\end{cases} \tag{11}$$

with H_i and K_i admissible potentials, H_1 and H_2 satisfying (MS), and furthermore

(GL) H_i and K_i are globally Lipschitz on $\mathbb{R}^d$, $i=1,2$,

(RK) ∇K_1 and ∇K_2 are continuous on $\mathbb{R}^2$.

Definition 2. *A curve $\boldsymbol{\mu}(\cdot)=(\mu_1(\cdot),\mu_2(\cdot)):[0,+\infty)\to\mathcal{P}_2(\mathbb{R}^d)^2$ is a* weak measure solution *to* (11) *is, for all $\phi,\psi\in C_c^\infty(\mathbb{R}^d)$, we have*

$$\begin{aligned}\frac{d}{dt}\int\phi(x)d\mu_1(x,t) &= -\frac{1}{2}\iint\nabla H_1(x-y)\cdot(\nabla\phi(x)-\nabla\phi(y))d\mu_1(x)d\mu_1(y)\\ &\quad-\iint\nabla K_1(x-y)\cdot\nabla\phi(x)d\mu_1(x)d\mu_2(y)\\ \frac{d}{dt}\int\psi(x)d\mu_2(x,t) &= -\frac{1}{2}\iint\nabla H_2(x-y)\cdot(\nabla\psi(x)-\nabla\psi(y))d\mu_2(x)d\mu_2(y)\\ &\quad-\iint\nabla K_2(x-y)\cdot\nabla\psi(x)d\mu_2(x)d\mu_1(y).\end{aligned}$$

IMPLICIT-EXPLICIT EULER SCHEME:

Let $\boldsymbol{\nu}$ be a reference measure, time independent and consider the relative energy functional:

$$\mathcal{F}[\boldsymbol{\mu}|\boldsymbol{\nu}] = \frac{1}{2}\int_{\mathbb{R}}(H_1*\mu_1 + K_1*\nu_2)\,d\mu_1 + \frac{1}{2}\int_{\mathbb{R}}(H_2*\mu_2+K_2*\nu_1)\,d\mu_2$$

We now construct the following implicit-explicit JKO scheme recursively. Let $\tau>0$ be a fixed time step, and let $\boldsymbol{\mu}_0=(\mu_{0,1},\mu_{0,2})\in\mathcal{P}(\mathbb{R}^d)^2$ be a fixed initial pair of probability measures. For a given $\boldsymbol{\mu}_\tau^n\in\mathcal{P}_2(\mathbb{R}^d)^2$, we define the sequence $\boldsymbol{\mu}_\tau^{n+1}$ as

$$\boldsymbol{\mu}_\tau^{n+1}\in\mathrm{argmin}_{\boldsymbol{\mu}\in\mathcal{P}_2(\mathbb{R}^d)^2}\left\{\frac{1}{2\tau}\mathcal{W}_2^2(\boldsymbol{\mu}_\tau^n,\boldsymbol{\mu})+\mathcal{F}\left[\boldsymbol{\mu}|\boldsymbol{\mu}_\tau^n\right]\right\}.$$

For a given choice of the sequence $\boldsymbol{\mu}_\tau^n=(\mu_{1,\tau}^n,\mu_{2,\tau}^n)$, we introduce the piecewise constant interpolation $\bar\mu_{i,\tau}(t)=\mu_{i,\tau}^n$, $t\in((n-1)\tau,n\tau]$ for $i=1,2$ and we define $\bar{\boldsymbol{\mu}}_\tau(t)=(\bar\mu_{1,\tau}(t),\bar\mu_{2,\tau}(t))$.

Theorem 3. *Let $T>0$. There exists an absolutely continuous curve $\bar{\boldsymbol{\mu}}:[0,T]\to\mathcal{P}_2(\mathbb{R}^d)^2$ such that the family $\bar{\boldsymbol{\mu}}_\tau(t)$ (up to a converging subsequence) satisfies $\bar{\boldsymbol{\mu}}_\tau\to\bar{\boldsymbol{\mu}}$ as $\tau\searrow 0$ uniformly on $[0,T]$.*

Theorem 4. *Let $\boldsymbol{\mu}_0\in\mathcal{P}_2(\mathbb{R}^d)^2$ be fixed. There exists an absolutely continuous curve $\boldsymbol{\mu}(\cdot):[0,+\infty)\to\mathcal{P}_2(\mathbb{R}^d)^2$ such that $\boldsymbol{\mu}(0)=\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}(t)$ is a weak measure solution to* (11) *in the sense of Definition 2. Such solution can be constructed as the limit (up to subsequences) of the approximating curve $\bar{\boldsymbol{\mu}}_\tau$.*

UNIQUENESS FOR SMOOTH KERNELS:

We use basically a bootstrap version of the characteristic method.

Theorem 5. *Assume that all the kernels H_i, K_i are C^2 and consider two initial measures $\boldsymbol{\mu}_0,\boldsymbol{\nu}_0\in\mathcal{P}_2(\mathbb{R}^d)^2$ with compact support and the related weak measure solutions of* (11) *$\boldsymbol{\mu},\boldsymbol{\nu}$ respectively. Then, there exists a constant $\tilde C>0$ such that*

$$\mathcal{W}_2(\boldsymbol{\mu}_t,\boldsymbol{\nu}_t)\le e^{\tilde Ct}\mathcal{W}_2(\boldsymbol{\mu}_0,\boldsymbol{\nu}_0)\quad t\ge 0. \tag{12}$$

Consequently, for a given initial condition $\boldsymbol{\mu}_0\in\mathcal{P}(\mathbb{R}^d)^2$, there exists a unique weak measure solution to (11).

NON-SYMMETRIZABLE SYSTEMS: A PREDETOR-PREY MODEL

Consider the system in one space dimension

$$\begin{cases}\partial_t\mu_1 = \partial_x(\mu_1(\partial_x S_1*\mu_1 + \partial_x K*\mu_2))\\ \partial_t\mu_2 = \partial_x(\mu_2(\partial_x S_2*\mu_2 - \alpha\partial_x K*\mu_1)),\end{cases} \tag{13}$$

Let K be an attractive potential. Under these assumptions, the system can be seen as a predator-prey system type, in which the first species is attracted by the second, which tries to escape.

We establish a criterion under which sums of Dirac's deltas

$$(\bar\mu_1,\bar\mu_2) = \left(\sum_{k=1}^N\rho_1^k\delta_{\bar X_k}(x),\sum_{h=1}^M\rho_2^k\delta_{\bar Y_h}(x)\right), \tag{14}$$

are stationary states for (13).

PARTICLES SYSTEM STABILITY:

We start our analysis with the particles system associated to (13)

$$\begin{cases}\dot X_i(t) = \displaystyle\sum_{k=1}^N m_X^k S_1'(X_k(t)-X_i(t)) + \sum_{h=1}^M m_Y^h K'(Y_h(t)-X_i(t))\\ \dot Y_j(t) = \displaystyle\sum_{h=1}^M m_Y^h S_2'(Y_h(t)-Y_j(t)) - \alpha\sum_{k=1}^N m_X^k K'(X_k(t)-Y_j(t)).\end{cases} \tag{15}$$

Let $\bar\Omega = \left(\bar X_1,...,\bar X_N,\bar Y_1,...,\bar Y_M\right)\in\mathbb{R}^{N+M}$ be a steady states for (15). Introducing the following quantities

$$d_X^i = \left(\sum_{k=1}^N m_X^k S_1''(\bar X_k-\bar X_i) + \sum_{h=1}^M m_Y^h K''(\bar Y_h-\bar X_i)\right)\qquad i=1,..,N$$

$$d_Y^j = \left(\sum_{h=1}^M m_Y^h S_2''(\bar Y_h-\bar Y_j) - \alpha\sum_{k=1}^N m_X^k K''(\bar X_k-\bar Y_j)\right)\qquad j=1,..,M,$$

$$\bar S_1 = \left(m_X^k S_1''(\bar X_k-\bar X_i)\right)_{i,k},\qquad \bar S_2 = \left(m_Y^h S_2''(\bar Y_h-\bar Y_j)\right)_{i,k},$$

$$\bar K_X = \left(m_Y^h K''(\bar Y_h-\bar X_i)\right)_{i,k},\qquad \bar K_Y = \left(m_X^k K''(\bar X_k-\bar Y_j)\right)_{i,k},$$

the linearised equation for (15) around $\bar\Omega$, with $\Omega(t)=\bar\Omega+\delta\Omega(t)$ is

$$\frac{d}{dt}\delta\Omega = (D+H)\,\delta\Omega,$$

with, for $i=2,...,N$ and $j=1,...,M$

$$H = \begin{pmatrix}\bar S_1 & \bar K_X\\ -\alpha\bar K_Y & \bar S_2\end{pmatrix}\quad D = \begin{pmatrix}\mathrm{diag}\left(-d_X^i\right) & 0\\ 0 & \mathrm{diag}\left(-d_Y^j\right)\end{pmatrix}. \tag{16}$$

So the system (15) is stable if and only if the matrix $D+H$ has strictly negative spectrum.

LOCAL NON-LINEAR STABILITY:

We rewrite (13) in the pseudo-inverse formalism,

$$\begin{cases}\partial_t u_1(z,t) = \int_0^1 S_1'(u_1(\zeta,t)-u_1(z,t))d\zeta + \int_0^1 K'(u_2(\zeta,t)-u_1(z,t))d\zeta\\ \partial_t u_2(z,t) = \int_0^1 S_2'(u_2(\zeta,t)-u_2(z,t))d\zeta - \alpha\int_0^1 K'(u_1(\zeta,t)-u_2(z,t))d\zeta,\end{cases} \tag{17}$$

with u_i, $i=1,2$ non-decreasing functions. Sums of Dirac's deltas corresponds to sums of increasing steps functions

$$(\bar u_1(z),\bar u_2(z)) = \left(\sum_{i=1}^N\bar X_i\chi_{I_1^i}(z),\sum_{h=1}^M\bar Y_h\chi_{I_2^h}(z)\right),\qquad I_l^p = \left[\sum_{j<p}m_l^j,\sum_{j\le p}m_l^j\right) \tag{18}$$

with $|I_l^p| = m_l^p$.

Theorem 6. *Let S_1, S_2 and K admissible and smooth potentials. Consider a steady state $(\bar u_1,\bar u_2)$ as in* (18) *for* (17)*, that satisfy:*

(NS1) d_X^h, d_Y^k are strictly positive for all $h=1,...,N$ and $k=1,...,M$;

(NS2) the matrix $H+D$ defined in (16) *has strictly positive spectrum, i.e. for some $\nu>0$, $\sigma(M)\subset\{z\in\mathbb{C}|\mathcal{R}](z)>\nu>0\}$.*

Then for all initial data $(u_{1,0},u_{2,0})$ such that, for $\epsilon>0$,

$$||u_{1,0}-\bar u_1||_\infty + ||u_{2,0}-\bar u_2||_\infty<\epsilon$$

exist a constant $C>0$ such that for all $t>0$

$$||u_1(t)-\bar u_1||_\infty + ||u_2(t)-\bar u_2||_\infty < C\left(1+t^{n-1}\right)e^{-\eta t}$$

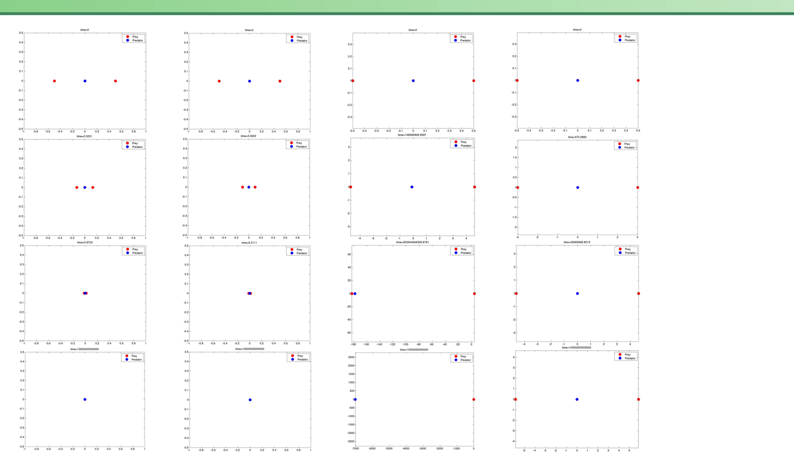
for some $\eta>0$.

ONE DIMENSIONAL PARTICLE SIMULATIONS:

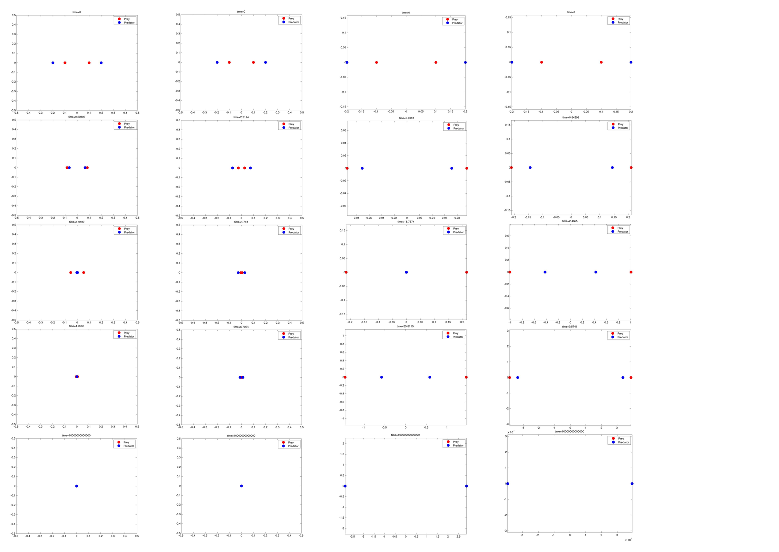
The following simulations are performed with normalized Gaussian cross-interaction potential and self-interaction potentials given by $S_1(x)=\frac{\beta_1}{2}(1-e^{-x^2})$ and $S_2(x)=\frac{\beta_2}{2}(1-e^{-x^2})$ respectively.

Consider a system with one predator and two prey. Calling $A=\frac{1}{2}K''(0)$ and $B_i=\frac{1}{2}S_i''(0)$, with $B_2=\frac{1}{2}\beta_2$, we obtain the following conditions on α:

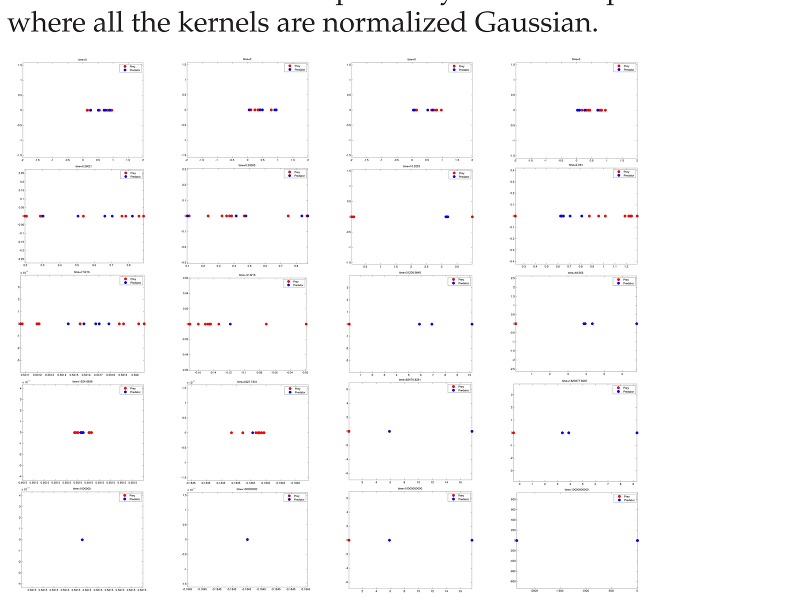
$$\alpha<1\qquad\alpha<\frac{B_2}{A}.$$



Taking into account now the case of two particle per species and setting $\frac{1}{2}K''(0)=A>0$, $B_1=\frac{1}{2}S_1''(0)$ and $B_2=\frac{1}{2}S_2''(0)$ we have $A+B_1>0$, $\alpha<\frac{B_2}{A}$, $\alpha<1$.

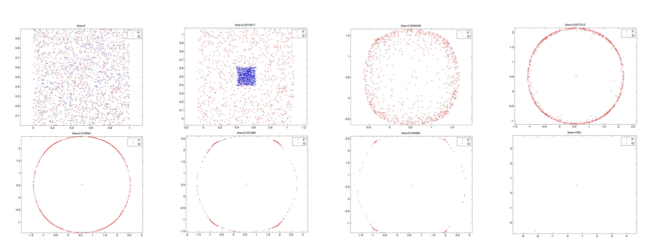


The last one-dimensional example is a system with 5 predator and 10 prey, where all the kernels are normalized Gaussian.

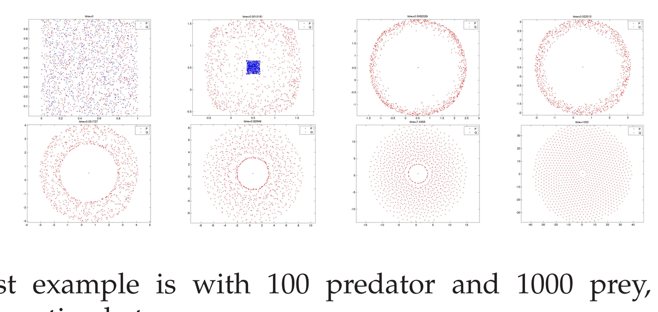


TWO DIMENSIONAL PARTICLE SIMULATIONS:

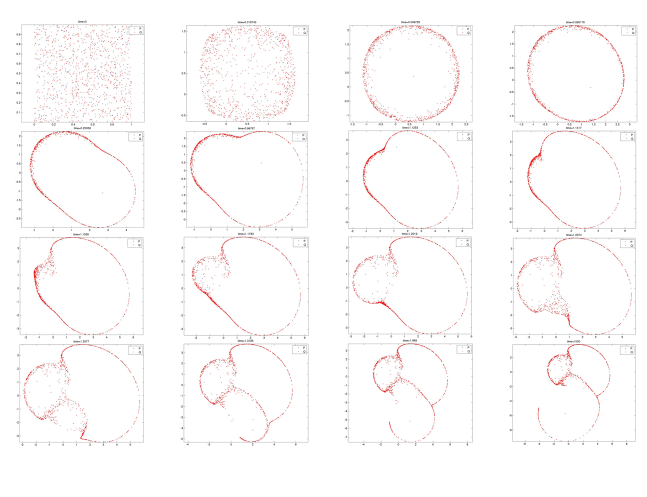
Here we present some simulation in the two-dimensional case. First we consider 1000 particles for species with attractive normalized Gaussian self-interaction potentials and $\alpha=1$.



The following dynamics is obtained with self-repulsive prey and self-attractive predator, with $\alpha=1$. All the kernels are normalized Gaussian.



The last example is with 100 predator and 1000 prey, with no self-interaction between prey.



Note that predators collapse into the center of mass and then start to chase the prey.

UPCOMING RESEARCH

- Stability analysis in space dimensions $d>1$;
- Numerical simulations for the PDE system