Stability of self-propelled particle systems and existence of flocking solutions for the continuum limit

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joint work with:
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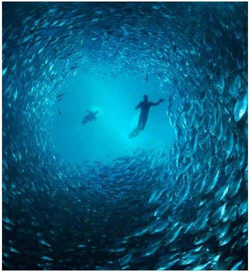
Motivation: Collective animal behaviour

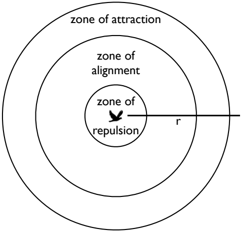
- Self-organization from local interaction, in absence of leadership
- Even simple interaction rules, reproducing patterns observed in nature
- Diversity of patterns, varying biological mechanisms competing with each other
- Mathematical challenge:
  $\longrightarrow$ Formulation of simplified models reproducing formations
  $\longrightarrow$ Pattern shape(s), convergence, stability, phase transitions, ...



Microscopic level Agent/particle model	$\longleftrightarrow$	Macroscopic level PDEs

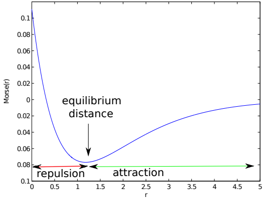
Self-propelled second-order interacting particle model

→ Three-zone approach to interaction potential



Modelling framework

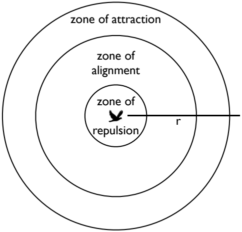
No alignment $\longrightarrow$



Interaction potential

Self-propelled second-order interacting particle model

$\rightarrow$ Three-zone approach to interaction potential



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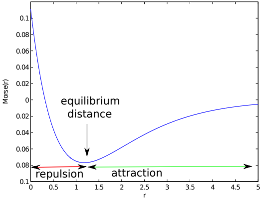

Modelling framework

Interaction potential

$\rightarrow$ The microscopic model for N particles, $(x_i, v_i) \in \mathbf{R}^d \times \mathbf{R}^d$

$$\frac{dx_i}{dt} = v_i \,,$$
$$\frac{dv_i}{dt} = \alpha v_i - \beta v_i |v_i|^2 - \frac{1}{N} \nabla_{x_i} \sum_{j \neq i} W(x_i - x_j)\,.$$

α, β: propulsion & friction force ; $W(x)$: interaction potential,

Self-propelled second-order interacting particle model

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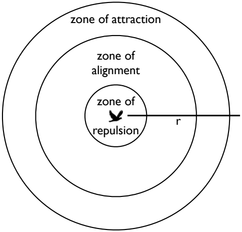

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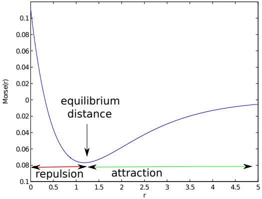

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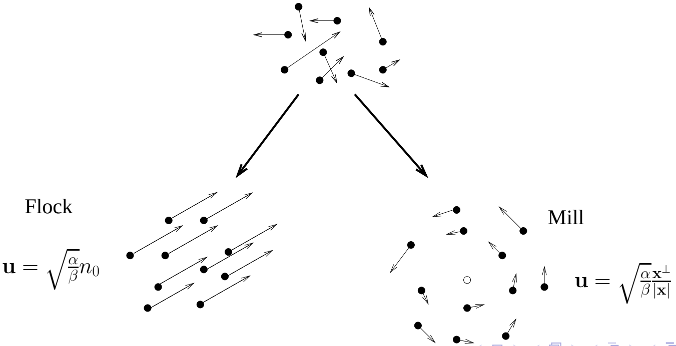
α, β: propulsion & friction force ; $W(x)$: interaction potential,

→ Other popular models: Cucker-Smale, Couzin-Viscek

Two common patterns: flock and mill

$$\frac{dx_i}{dt} = v_i\,,$$
$$\frac{dv_i}{dt} = \underbrace{\alpha v_i - \beta v_i|v_i|^2}_{\text{speed } \sqrt{\alpha/\beta}} - \underbrace{\frac{1}{N}\nabla_{x_i}\sum_{j\neq i} W(x_i - x_j)}_{\text{the spatial shape}}\,.$$



Flocks and mills

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Two basic questions in this talk:

- Simpler (reduced) system to look for stable patterns?
- Existence of patterns in the parameter regimes?

Stability for flocks: from second order to first order

$$\frac{dx_i}{dt} = v_i\,,$$
$$\frac{dv_i}{dt} = \alpha v_i - \beta v_i|v_i|^2 - \frac{1}{N}\nabla_{x_i}\sum_{j\neq i} W(x_i - x_j)\,. \qquad \text{(2nd)}$$

Flocking solution: $\big(x_i(t), v_i(t)\big) = (\widehat{x}_i + m_0 t, m_0)$

- the constant mean velocity m_0 with speed $\sqrt{\frac{\alpha}{\beta}}$
- the spatial configurations $\{\widehat{x}_i\}$ satisfy $\nabla_{\widehat{x}_i}\sum_{j\neq i} W(\widehat{x}_i - \widehat{x}_j) = 0,\ i = 1, 2, \cdots, N$

The condition for the spatial configurations motivates the first order system

$$\frac{dx_i}{dt} = -\frac{1}{N}\nabla_{x_i}\sum_{j\neq i} W(x_i - x_j). \qquad \text{(1st)}$$

Linearized system: first order

$$\frac{dx_i}{dt} = -\frac{1}{N}\nabla_{x_i}\sum_{j\neq i} W(x_i - x_j), \tag{1st}$$

Assuming $x_i(t) = \widehat{x}_i + \widehat{\delta x}_i(t)$, the linearized system for the perturbation $\widehat{\delta x} = (\widehat{\delta x}_1, \cdots, \widehat{\delta x}_N)$ is

$$\frac{d}{dt}\widehat{\delta x} = G(\widehat{x})\widehat{\delta x},$$

where

$$G_{ij} = \begin{cases} -\frac{1}{N}\sum_{j\neq i} \text{Hess } W(\hat{x}_i - \hat{x}_j) & \text{for } i = j \\ \frac{1}{N}\text{Hess } W(\hat{x}_i - \hat{x}_j) & \text{for } i \neq j \end{cases},$$

and Hess W is the Hessian matrix of W.

Linearized first order system: Eigenvalues of $G(\widehat{x})$

The spatial configurations at $\widehat{x} = (\widehat{x}_1, \cdots, \widehat{x}_N)$ are stable if $G(\widehat{x})$ has

- no positive eigenvalues and
- no generalized eigenvectors for eigenvalue zero.

The simplest unstable system with zero eigenvalues:

$$\frac{d}{dt} x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$$

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Eigen-structures of the Jacobian $G(\widehat{x})$:

a) $2d - 1$ zero eigenvalues: translations (d) and rotations ($d - 1$).

b) no generalized eigenvector for zero eigenvalue ($G(\widehat{x})$ is derived from a potential and symmetric)

c) $G(\widehat{x})$ is non-positive semi-definite

Linearized system: second order

$$
\begin{aligned}
\frac{dx_i}{dt} &= v_i\,,\\
\frac{dv_i}{dt} &= \alpha v_i - \beta v_i|v_i|^2 - \frac{1}{N}\nabla_{x_i}\sum_{j\neq i} W(x_i - x_j)\,.
\end{aligned}
\tag{2nd}
$$

For solutions with mean velocity m_0, assuming

$$x_i(t) = \widehat{x}_i + tm_0 + \widehat{\delta x}_i(t), \quad v_i(t) = m_0 + \widehat{\delta v}_i,$$

the linearized system is

$$\frac{d}{dt}\widehat{\delta x}_i = \widehat{\delta v}_i, \qquad \frac{d}{dt}\widehat{\delta v}_i = \sum G_{ij}(\widehat{x})\delta x_j - 2\beta(m_0\cdot\widehat{\delta v}_i)m_0,$$

or

$$\frac{d}{dt}\begin{pmatrix}\widehat{\delta x}\\ \widehat{\delta v}\end{pmatrix} = \begin{pmatrix} O & Id \\ G(\widehat{x}) & -2\beta \mathrm{kron}(Id, m_0\oplus m_0)\end{pmatrix}\begin{pmatrix}\widehat{\delta x}\\ \widehat{\delta v}\end{pmatrix}.$$

Eigen-structure for the linearized second order system

If $(\widehat{\delta x}, \widehat{\delta v})$ is an eigenvector with eigenvalue λ, then

$$\lambda\widehat{\delta x} = \widehat{\delta v}, \quad \lambda\widehat{\delta v} = G(\widehat{x})\widehat{\delta x} - 2\beta \text{kron}(Id, m_0 \oplus m_0)\widehat{\delta v},$$

or the "reduced eigenvalue problem"

$$\lambda^2\widehat{\delta x} = G(\widehat{x})\widehat{\delta x} - 2\lambda\beta \text{kron}(Id, m_0 \oplus m_0)\widehat{\delta x}.$$

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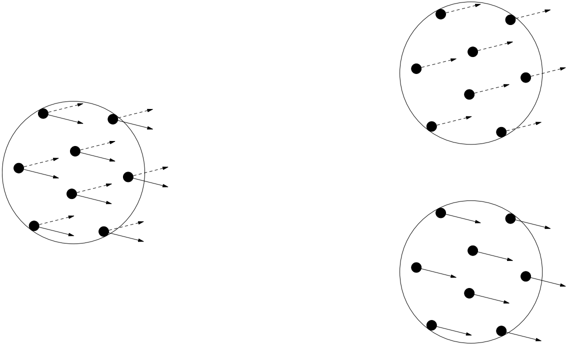
Taking the inner problem of the equation with $\widehat{\delta x}$,

$$\langle \widehat{\delta x}, \widehat{\delta x} \rangle \lambda^2 + \underbrace{2\beta \sum_i \langle m_0, \widehat{\delta x}_i \rangle^2}_{\geq 0} \lambda - \underbrace{\langle \widehat{\delta x}, G(\widehat{x})\widehat{\delta x} \rangle}_{\leq 0} = 0.$$

$\Longrightarrow \lambda \leq 0!$

Generalized eigenvectors for linearized second order system

The perturbation $\widehat{\delta v}_i = \widehat{\delta v}$ (the same for each $\widehat{v}_i$) leads to another valid flock, but not "stable" in the previous context.



The speed is kept the same $\Longrightarrow$ make 90^o turn, $\widehat{\delta v} = m_0^{\perp}$

Generalized eigenvectors for linearized second order system

The generalized vector for at eigenvalue zero:

$$\widehat{\delta x}_i = 0, \quad \widehat{\delta v}_i = m_0^{\perp}.$$

Simple linear algebra:

$$\begin{pmatrix} O & Id \\ G(\widehat{x}) & -2\beta \text{kron}(Id, m_0 \oplus m_0) \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \text{kron}(\mathbf{1}, m_0^{\perp}) \end{pmatrix} = \begin{pmatrix} \text{kron}(\mathbf{1}, m_0^{\perp}) \\ \mathbf{0} \end{pmatrix}$$

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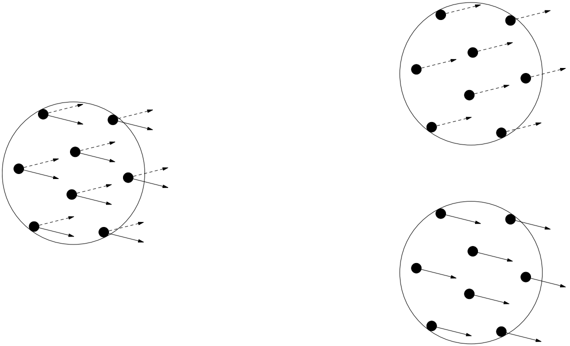
Generalized eigenvectors for linearized second order system

Something wrong about the linearization $x_i(t) = \widehat{x}_i + t m_0 + \widehat{\delta x}_i(t)$?



$\Longrightarrow$ introduce the "real time" mean velocity $m(t) = \frac{1}{N} \sum_i v_i(t)$ and use the new linearization

$$x_i(t) = \widehat{x}_i + \int_0^t m(s) ds + \widehat{\delta x}_i(t),$$

Stability in the new settings

The equation for the mean velocity:

$$\frac{d}{dt}m = \frac{1}{N}\sum_i \frac{dv_i}{dt} = \frac{1}{N}\sum_i(\alpha v_i - \beta|v_i|^2 v_i).$$

A few remarks:

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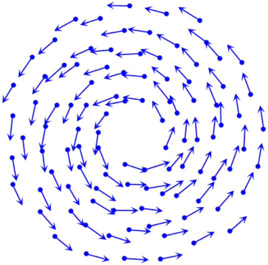
d) True stability theorem for flocks: the flock is a normally hyperbolic invariant manifold (parametrized by $\widehat{x}$ and m_0)

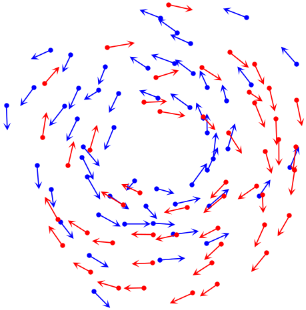
e) Spectral gap is shrinking (the first negative eigenvalue approaches zero as N increases)

What about the stability of the rotating mills?

The equation to find the steady configurations

$$\frac{d}{dt}x_i = -\frac{1}{N}\nabla_{x_i}\sum_{j\neq i} W(x_i - x_j) + \sqrt{\frac{\alpha}{\beta}}\frac{x_i}{|x_i|^2}.$$





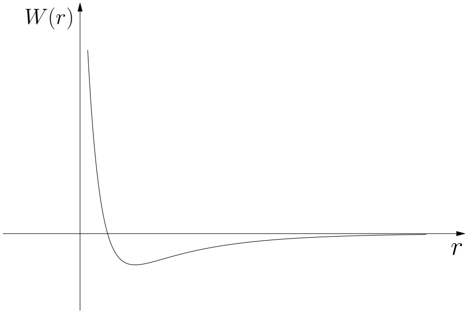
Existence of (flock) patterns for the particle system

When is there a concurrent moving flock?

$$\frac{dx_i}{dt} = v_i\,,$$
$$\frac{dv_i}{dt} = \alpha v_i - \beta v_i |v_i|^2 - \frac{1}{N} \nabla_{x_i} \sum_{j \neq i} W(x_i - x_j)\,. \qquad \text{(2nd)}$$

"micro" and "macro" conditions for flocks

a) For two particles, W should be biologically relevant:



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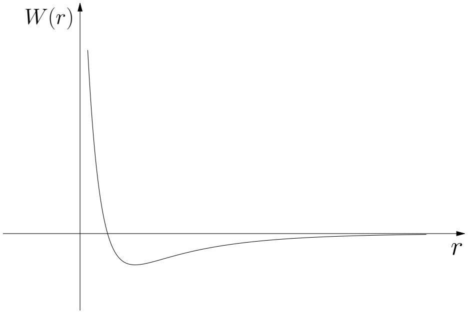


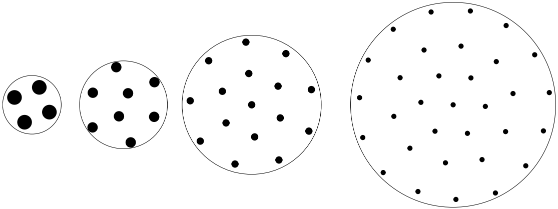
b) As more “averaged particles” are added into the system, a continuum density is approached:

W is not H-stable.

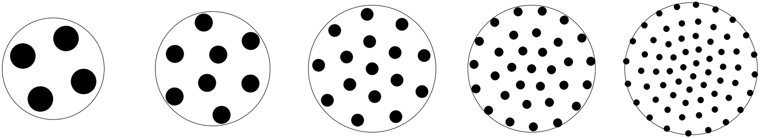
H-stable vs Catastrophic potential W

Different behaviours of the minimizers of $\sum_{i,j}^{N} W(x_i - x_j)$

H-stable potential: minimal distance between particles are approximately the same, forming crystal-like structures



Catastrophic (non H-stable) potential: the total size does not expand



H-stable vs Catastrophic potential W in statistical mechanics

A potential W for a many-body system is called **H-stable** (or simply **stable**) if the potential energy per particle is bounded below by a constant that is independent of the total number of particles, i.e.,

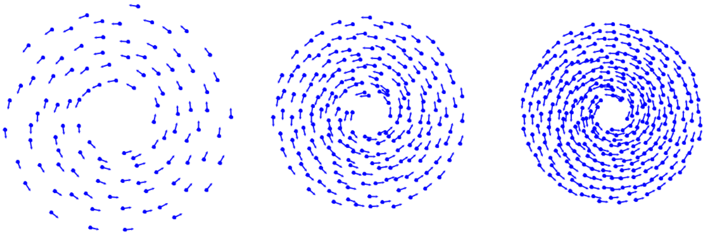
$$\sum_{i,j}^{N} W(x_i - x_j) > -NB.$$

If the positions $x_1, x_2, \cdots, x_N \in \mathbb{R}^n$ minimize the interaction energy $\sum_{i,j}^{N} W(x_i - x_j)$, then

- If W is **H-stable**, $\sum_{i,j}^{N} W(x_i - x_j) = O(N)$
- If W is **catastrophic**, $\sum_{i,j}^{N} W(x_i - x_j) = O(N^2)$

Why the potentials are called “catastrophic”?

$$\frac{dx_i}{dt} = v_i \,,$$
$$\frac{dv_i}{dt} = \alpha v_i - \beta v_i |v_i|^2 - \nabla_{x_i} \sum_{j \neq i} W(x_i - x_j)\,.$$



$N = 100, 200, 300$. Parameters [1]: $W(x) = C_r e^{-|x|/\ell_r} - C_a e^{-|x|/\ell_a}$, $C_a = 0.5, C_r = 1.0, \ell_a = 2.0, \ell_r = 0.5, \alpha = 1.6, \beta = 0.5$

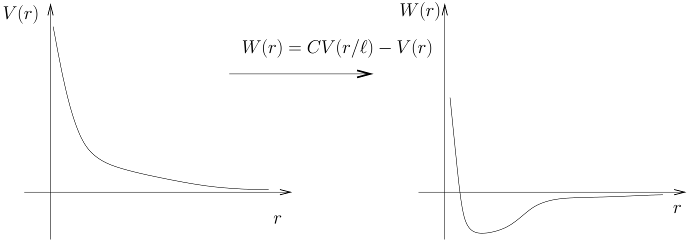
[1] M. R. D'Orsogna et al, PRL 96, 104302 (2006)

Catastrophic potential: the definition used in the talk

Conditions for W to be catastrophic:

$$\int_{\mathbf{R}^d} W < 0.$$

If $W(r)$ is constructed from a decreasing function $V(r)$,



then $\int_{\mathbf{R}^d} W < 0$ becomes $(C\ell^d - 1)\int_{\mathbf{R}^d} V < 0$ or

$$C\ell^d - 1 < 0.$$

From particle system to hydrodynamic equations

$$\frac{dx_i}{dt} = v_i \,,$$
$$\frac{dv_i}{dt} = \alpha v_i - \beta v_i |v_i|^2 - \frac{1}{N} \nabla_{x_i} \sum_{j \neq i} W(x_i - x_j) \,.$$

- The kinetic equation for $f(t, x, v)$ in the mean field limit:

$$\partial_t f + v \cdot \nabla_x f + F[\rho] \cdot \nabla_v f + \mathrm{div}_v \left(\left(\alpha - \beta |v|^2 \right) v f \right) = 0 \,,$$

$\rho(t, x) = \int f(t, x, v) \, \mathrm{d}v$: macroscopic density;

$F[\rho] = -\nabla_x W \star \rho$: interaction force

- Mono-kinetic ansatz: $f(t, x, v) = \rho(t, x) \delta(v - \mathbf{u}(t, x))$,

$$\begin{cases} \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = 0, \\ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = (\alpha - \beta |\mathbf{u}|^2) \mathbf{u} - \nabla_x W \star \rho. \end{cases}$$

- For flocks: $\mathbf{u}(t, x) = \mathbf{u}_0 = \sqrt{\frac{\alpha}{\beta}} n_0$, $\rho(t, x) = \rho_F(x - \mathbf{u}_0 t)$, where ρ_F satisfies $\nabla_x W \star \rho_F = 0$.

Energy, gradient flow

The energies:

$$\frac{1}{2N}\sum_{i,j} W(x_i - x_j) \quad \text{or} \quad F(\rho) = \frac{1}{2}\int \rho W * \rho$$

The corresponding gradient flow systems:

$$\frac{dx_i}{dt} = -\frac{1}{N}\nabla_{x_i}\sum_{j\neq i} W(x_i - x_j) \quad \text{or} \quad \rho_t = \nabla\cdot(\rho\nabla W * \rho) = \nabla\cdot(\rho\nabla\frac{\delta F}{\delta\rho}).$$

The steady state equation for the continuum equation:

$$W * \rho = D.$$

Yet another non H-stable condition

non H-stable condition in Jose's talk:

a) $\lim_{r\to\infty} W(r) = 0$, $W(0)$ is finite

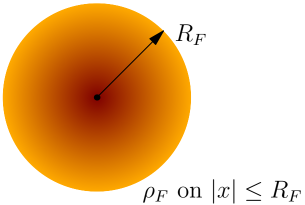
b) There is ρ such that $F(\rho) < 0$

How is this definition related to other definitions?

- If there is a such ρ, approximate $\rho(x)$ by $\frac{1}{N}\sum_j \delta(x - x_i) \Longrightarrow \sum_{i,j} W(x_i - x_j) = O(-N^2)$;
- if there is no such ρ, $\rho(t)$ spreads to infinity.

The subtleties in **normalization** by $1/N$ and **self-energy** $\sum_i^N W(0)$

Governing equations for the (radial) flock profile ρ_F



The spatial profile ρ_F of the flock satisfies

$$\text{Flock}: \quad W \star \rho_F = D_F \qquad \text{in } B(0, R_F) = \text{supp}(\rho_F)$$

Rigorous existence results for Quasi-Morse potentials [2]

- Idea: convert integral equations into differential equations
- $W(x) = V(|x|) - CV(|x|/\ell)$ (simple condition for W to be non H-stable)
- Desired properties on $V(r)$: non-negative, fast decay to zero $-->$ Fundamental solutions of the operator $\Delta - \mathrm{Id}$.
- Dimension-dependent potential

$$\begin{cases} d = 1: & V(r) = -\frac{1}{2}e^{-r} \\ d = 2: & V(r) = -\frac{1}{2\pi}K_0(r) \\ d = 3: & V(r) = -\frac{1}{4\pi}\frac{e^{-r}}{r} \end{cases}$$

  and in general dimension n:

$$V(r) = -(2\pi)^{-\frac{d}{2}} r^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(r).$$

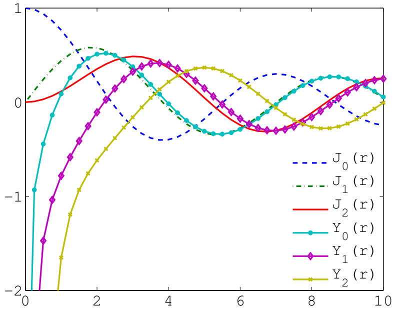
- Price to pay: working with less familiar (modified) Bessel functions.

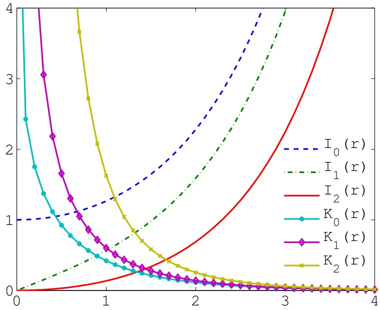
[2] J.A.Carrillo, S.Martin and V. Panferov, Physica D 2013

What are the (modified) Bessel functions?

Solutions to (for $J_\nu(r)$ and $Y_\nu(r)$) $r^2\dfrac{d^2y}{dr^2}+r\dfrac{dy}{dr}+(r^2-\nu^2)y=0$

or (for $I_\nu(r)$ and $K_\nu(r)$) $r^2\dfrac{d^2y}{dr^2}+r\dfrac{dy}{dr}-(r^2+\nu^2)y=0.$





From integral equation to differential equation

If $W(x) = V(x) - CV(x/\ell)$ with $\Delta V(x) - V(x) = \delta_0$, then

$$\boxed{\left(\Delta - \mathrm{Id}\right)\left(\Delta - \frac{1}{\ell^2}\,\mathrm{Id}\right) W \star \rho = (1 - C)\Big[\Delta\rho + A\rho\Big] = \tilde{D},}$$

with $A = (1 - C\ell^d)/(C\ell^d - \ell^2)$.
Since ρ satisfies the ODE

$$\frac{d^2\rho}{dr^2} + \frac{d-1}{r}\frac{d\rho}{dr} + A\rho = \tilde{D},$$

for $r \leq R_F$

$$\rho(r) = \begin{cases} \mu_1 r^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(ar) + \mu_2, & A > 0 \\ \mu_1 r^2 + \mu_2, & A = 0, \\ \mu_1 r^{1-\frac{d}{2}} I_{\frac{d}{2}-1}(ar) + \mu_2, & A < 0. \end{cases}$$

with $a = \sqrt{|A|} \implies W * \rho$ is some function depends on μ_1 and μ_2.

Another approach for $W * \rho$

Since $W * \rho$ satisfies the fourth order ODE

$$\boxed{\Big(\Delta - \mathrm{Id}\Big)\Big(\Delta - \frac{1}{\ell^2}\,\mathrm{Id}\Big) W \star \rho = \tilde{D},}$$

with $A = (1 - C\ell^d)/(C\ell^d - \ell^2)$,

$$W \star \rho(r) = \tilde{D} + \lambda_1 r^{1-d/2} I_{\frac{d}{2}-1}(r) + \lambda_2 r^{1-d/2} I_{\frac{d}{2}-1}(r/\ell).$$

The solvability condition for R_F: $\lambda_1(\mu_1, \mu_2) = 0, \lambda_2(\mu_1, \mu_2) = 0$

Solvability condition for the flock profile $W \star \rho = D$

$W \star \rho$ is a constant on $[0, R]$ (in radial variable) only if

$$M\boldsymbol{\mu} = \begin{pmatrix} \tilde{B}(1) & 1 \\ \tilde{B}(\ell) & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or $\det M = \tilde{B}(1) - \tilde{B}(\ell) = 0$. Here $\tilde{B}(\xi)$ is defined as

$$\tilde{B}_+(\ell) = R^{1-\frac{d}{2}} \left(1 + a^2\ell^2\right)^{-1} \left[J_{\frac{d}{2}-1}(aR) \frac{K_{\frac{d}{2}-2}(R/\ell)}{K_{\frac{d}{2}}(R/\ell)} + a\ell J_{\frac{d}{2}-2}(aR) \frac{K_{\frac{d}{2}-1}(R/\ell)}{K_{\frac{d}{2}}(R/\ell)} \right],$$

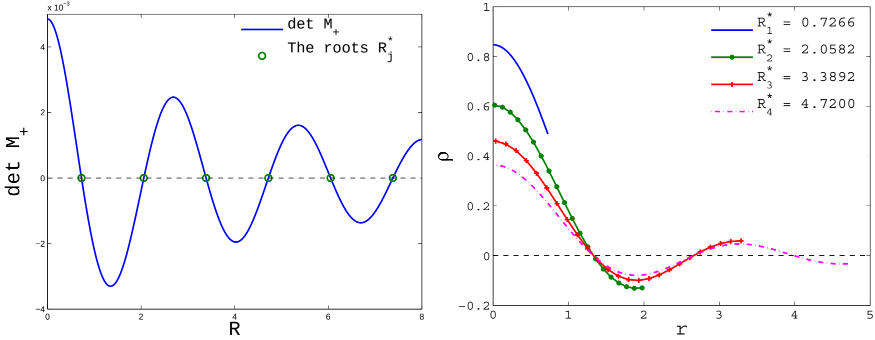
$$\tilde{B}_0(\ell) = 2\ell R \frac{K_{\frac{d}{2}+1}(R/\ell)}{K_{\frac{d}{2}}(R/\ell)} + 1,$$

$$\tilde{B}_-(\ell) = R^{1-\frac{d}{2}} \left(1 - a^2\ell^2\right)^{-1} \left[I_{\frac{d}{2}-1}(aR) \frac{K_{\frac{d}{2}-2}(R/\ell)}{K_{\frac{d}{2}}(R/\ell)} + a\ell I_{\frac{d}{2}-2}(aR) \frac{K_{\frac{d}{2}-1}(R/\ell)}{K_{\frac{d}{2}}(R/\ell)} \right],$$

where $a^2 = A = \left|\frac{1-C\ell^d}{C\ell^d - \ell^2}\right|$.

Theorem: $\det M = 0$ for some $R > 0$ if and only if $A > 0$.

Existence and uniqueness in 3D



The determinant $M_+ = \tilde{B}(1) - \tilde{B}(\ell)$ is oscillatory (as a function of R, the size of the support). There are infinite many roots for $M_+ = 0$, but only the first root gives a nonnegative density ρ_F.

Main Results for the flock profile governed by $W \star \rho = D$

Let W be a Quasi-Morse potential:

Theorem (Existence and uniqueness in 3D)

In space dimension $d = 3$ with parameters in the regime $C\ell > 1, \ell < 1$. Then flock profiles exist if and only if $A > 0$. Furthermore, if $A > 0$, there exists a unique flock profile.

Theorem (Existence in 2D)

In space dimension $d = 2$ with parameters in the regime $C > 1, \ell < 1$. Then flock profiles exist if and only if $A > 0$ or equivalently $C\ell^2 < 1$.

Why uniqueness in 3D?

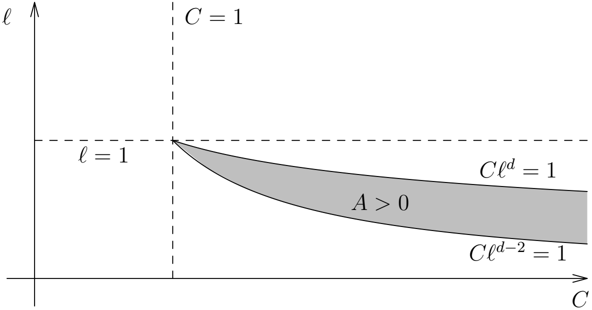
$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \; J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x, \; I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x,$$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x, \; K_{1/2}(x) = K_{-1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}.$$

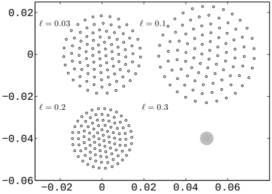
Collecting all the pieces

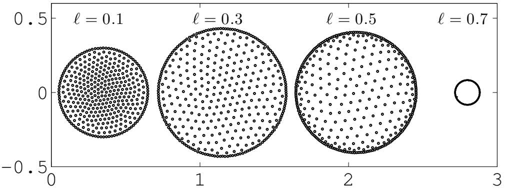
For the quasi-Morse potential $W(x) = V(x) - CV(x/\ell)$,

- Existence of equilibrium distance for **two particles**:
$$\ell < 1, C\ell^{d-2} > 1$$
- Conditions for the **non H-stability**:
$$\int W = (1 - C\ell^d) \int V < 0$$
- **Existence of nonnegative solutions** for $W \star \rho = D$:
$$A = \frac{1 - C\ell^d}{C\ell^d - \ell^2} > 0.$$

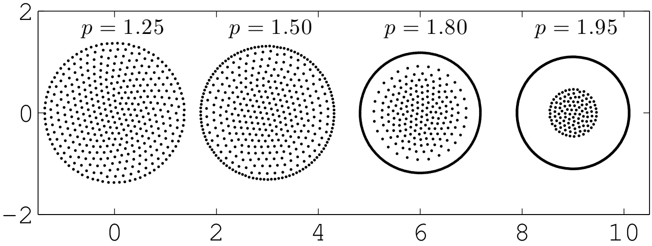


How about $W(x) = Ce^{-(r/\ell)^p} - e^{-r^p}$





$p = 1/2, C = 0.6$ $p = 3/2, C = 0.6$



$C = 10/9, \ell = 3/4$

Conclusion and open problems

- The existence of flocks in the parameter space (C, ℓ) for the particle system

$$\frac{dx_i}{dt} = v_i\,,$$
$$\frac{dv_i}{dt} = \alpha v_i - \beta v_i |v_i|^2 - \frac{1}{N}\nabla_{x_i} \sum_{j\neq i} W(x_i - x_j)\,.$$

  can be determined by the first order system and is numerically indicated by (1) $W(r)$ is stable for two particles; (2) W is not H-stable

- The parameter space is the same for the corresponding continuum integral equation $W \star \rho = D$, for Quasi-Morse potential.
- Generalization to integral equations $W \star \rho = D$ for other potentials W, like the Morse potential $W(x) = Ce^{-|x|/\ell} - e^{-|x|}$?

References

- J.A. Carrillo, S. Martin, V. Panferov. A new interaction potential for swarming models. *Physica D: Nonlinear Phenomena*, 260: 112-126, 2013.
- J.A. Carrillo, Y. Huang and S. Martin. Nonlinear stability of flock solutions in second-order swarming models. *Nonlinear Analysis: Real World Applications*, 17(0):332–343, 2014.
- J.A. Carrillo, Y. Huang and S. Martin. Explicit Flock Solutions for Quasi-Morse potentials. To appear in *European J. Appl. Math*. Preprint, arXiv:1312.0469, 2013.