Well-posedness of the Cauchy problem for nonlinear Kolmogorov-Fokker-Planck equations for measures

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We consider the Cauchy problem:

$$\partial_t \mu_t = \partial_{x_i x_j}(a^{ij}(x,t,\mu)\mu_t) - \partial_{x_i}(b^i(x,t,\mu)\mu_t), \quad \mu_0 = \nu, \tag{1}$$

where μ_t and ν are probability measures on $\mathbb{R}^d$.
Set $L_\mu u = a^{ij}(x,t,\mu)\partial_{x_i x_j} u + b^i(x,t,\mu)\partial_{x_i} u$.
Definition. $\mu = (\mu_t)_{t\in[0,\tau]}$ is a *solution* of (1) iff all coefficients $a^{ij}(x,t,\mu)$, $b^i(x,t,\mu)$ are μ-locally integrable and for all $t \in [0,\tau]$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$ the identity holds:

$$\int \varphi d\mu_t - \int \varphi d\nu = \int_0^t \int L_\mu \varphi d\mu_s ds.$$

Some examples

1. Transport equation (cf. [**?**, **?**, **?**, **?**]). Set $\dot{x}_t = b(x_t)$, $x_0 = x$. Then $\mu_t(B) = \nu(x_t^{-1}(B))$ satisfies

$$\partial_t \mu = -\operatorname{div}(b\mu_t), \quad \mu_0 = \nu.$$

2. Vlasov equation (cf. [**?**]). Let $\dot{x}_t^j = N^{-1}\sum_{i=1}^N B\left(x_t^j, x_t^i\right)$. Set $\mu_t = N^{-1}\sum \delta_{x_t^i}$. Then

$$\partial_t \mu_t = -\operatorname{div}(b\mu_t), \quad b(x,t,\mu) = \int B(x,y)\,d\mu_t.$$

3. Fokker-Planck-Kolmogorov equation (cf. [**?**, **?**, **?**]). Consider

$$dX_t = dW_t + b(X_t)\,dt, \quad X_0 = x.$$

Then $\mu_t(B) = \int P(X_t \in B)\,d\nu$ satisfies $\partial_t \mu_t = \triangle \mu_t - \operatorname{div}(b\mu_t)$, $\mu_0 = \nu$.

4. McKean-Vlasov equation (cf. [**?**]). Let $dX_t^{i,N} = dW_t^i + N^{-1}\sum_{j=1}^N B\left(X_t^j, X_t^i\right)$. Then $X_t^{i,N}$ converges in law to solution X_t of $dX_t = dW_t + b(X_t)\,dt$ and

$$\partial_t \mu_t = \triangle \mu_t - \operatorname{div}(b(\mu_t, x)\mu_t),$$

where $b(\mu_t, x) = \int B(x,y)d\mu_y(y)$, $\mu_t(dy) = P(X_t \in dy)$.
This result is called propagation of chaos for McKean-Vlasov equations.

Typical coefficients contain expressions like $\int K(x,y,t)\,d\mu_t$ or $\int_0^t \int K(x,y,s)\,d\mu_s\,ds$, with a kernel K growing at infinity.

Let $\tau_0 > 0$ be a fixed number and V be a nonnegative function. For each $\alpha \in C^+([0,\tau_0])$ we *denote* by $M_{\tau,\alpha}(V)$ the set of nonnegative measures $\mu = (\mu_t)_{t\in[0,\tau]}$ such that

$$\int V(x)\,d\mu_t \le \alpha(t) \quad \forall t \in [0,\tau].$$

In typical examples $V(x) = 1 + |x|^p$ or $\exp(k|x|^r)$.

Assumptions

(H1) There exists a **Lyapunov function**: $V \in C^2(\mathbb{R}^d)$:

$$V(x) > 0, \quad \lim_{|x|\to+\infty} V(x) = +\infty,$$

and mappings Λ_1 and Λ_2 of $C^+([0,\tau_0])$ into $C^+([0,\tau_0])$: functions a^{ij} and b^i are defined on each $M_{\tau,\alpha} = M_{\tau,\alpha}(V)$ and

$$L_\mu V(x,t) \le \Lambda_1[\alpha](t) + \Lambda_2[\alpha](t)V(x).$$

Remark. Typical examples of Λ_1 and Λ_2 are $\alpha(t) \mapsto G(\alpha(t))$ or $\alpha(t) \mapsto \int_0^t G(\alpha(s))\,ds$.
Definition. A sequence of measures $\mu^n = (\mu_t^n)_{t\in[0,\tau]}$ in $M_{\tau,\alpha}$ *V-converges* to a measure $\mu = (\mu_t)_{t\in[0,\tau]}$ in $M_{\tau,\alpha}$ if for all $t \in [0,\tau]$

$$\lim_{n\to\infty} \int F(x)d\mu_t^n = \int F(x)d\mu_t$$

for every $F \in C(\mathbb{R}^d)$: $\lim_{|x|\to\infty} F(x)/V(x) = 0$.

(H2) (continuity)

- for all $\tau \in (0,\tau_0]$, $\alpha \in C^+([0,\tau_0])$, $\sigma \in M_{\tau,\alpha}$, and $x \in \mathbb{R}^d$ the mappings

$$t \mapsto a^{ij}(x,t,\sigma), \quad t \mapsto b^i(x,t,\sigma)$$

  are Borel measurable on $[0,\tau]$
- the mappings $x \mapsto b^i(x,t,\sigma)$ and $x \mapsto a^{ij}(x,t,\sigma)$ are bounded on closed balles uniformly in $\sigma \in M_{\tau,\alpha}$ and $t \in [0,\tau]$ and continuous on closed balls uniformly in $\sigma \in M_{\tau,\alpha}$ and $t \in [0,\tau]$
- $\mu^n \in M_{\tau,\alpha}$ is V-convergent to $\mu \in M_{\tau,\alpha}$, then

$$\lim_{n\to\infty} a^{ij}(x,t,\mu^n) = a^{ij}(x,t,\mu), \qquad \lim_{n\to\infty} b^i(x,t,\mu^n) = b^i(x,t,\mu).$$

Example: The Assumption (H2) is fulfilled for

$$b(x,t,\mu) = \int K(x,y)d\mu_t$$

with a continuous vector field K on $\mathbb{R}^d \times [0,\tau]$: $|K(x,y)| \le C_1(x) + C_2(x)V^{1-\gamma}(y)$, $\gamma \in (0,1)$, and continuous $C_1(x)$, $C_2(x)$.
(H3) (parabolicity) $A(x,t,\sigma) = (a^{ij}(x,t,\sigma))_{1\le i,j\le d}$ is symmetric and nonnegative definite.

Main Existence Theorem

Suppose (H1), (H2), (H3), initial data ν is a probability measure on $\mathbb{R}^d$ and $V \in L^1(\nu)$. Then

(i) There exists $\tau \in (0,\tau_0]$ such that the Cauchy problem (1) has a solution on $[0,\tau]$.

(ii) If Λ_1 and Λ_2 are constant then the Cauchy problem (1) has a solution on the whole $[0,\tau_0]$.

(iii) Suppose $\Lambda_1[\alpha] = 0$ and $\Lambda_2[\alpha](t) = G(\alpha(t))$ for some strictly increasing continuous positive function G on $[0,+\infty)$. Then the Cauchy problem (1) has a solution on each $[0,\tau]$ with $\tau \in (0, \min\{T,\tau_0\}]$, where

$$T = \int_{u_0}^{+\infty} \frac{1}{uG(u)}\,du, \quad u_0 = \int V(x)\,d\nu.$$

In all cases μ_t are probability measures and $\sup_{t\in[0,\tau]} \int V(x)d\mu_t < \infty$.

The proof is based on Schauder fixed-point theorem and vanishing viscosity method (cf. [**?**]).
How does this theorem work?

An Important Example

$$\partial_t \mu_t = \operatorname{div}\Big(\mu_t \int |x-y|^{m-1}(x-y)\,d\mu_t\Big), \quad \mu_0 = \nu. \tag{2}$$

Let $m > 0$. Suppose ν is a probability measure on $\mathbb{R}^d$ and $|x|^{m+1} \in L^1(\nu)$ ($|x|^2 \in L^1(\nu)$ if $m < 1$). One can see that all assumptions of the previous theorem are fulfilled. Thus there exists $\tau > 0$ such that on $[0,\tau]$ the Cauchy problem (2) has a probability solution with uniformly bounded moments of order $m+1$. If $0 < m < 1$ the solution is global with uniformly bounded second moments.

Absence of global solution

Estimates for the existence time (provided in Main Existence Theorem) are in some sense exact.

"Blow-up" Theorem
$V \in C^2(\mathbb{R}^d)$, $V \ge 0$, $\lim_{|x|\to\infty} V(x) = +\infty$, G is a continuous positive increasing function on $[0,+\infty)$. The coefficients of the operator

$$L_\mu = a^{ij}(x,t,\mu)\partial_{x_i x_j} + b^i(x,t,\mu)\partial_{x_i}$$

are defined on every set $M_{\tau,\alpha}(V)$ and for all $\mu \in M_{\tau,\alpha}(V)$ and all $(x,t) \in \mathbb{R}^d \times [0,\tau]$ one has

$$L_\mu V(x,t) \ge G\Big(\int V(x)d\mu_t\Big)V(x).$$

Suppose that $|\sqrt{A(x,t,\mu)}\nabla V(x)|^2 \le C_1 + C_2 V(x)$ for some $C_1 > 0$ and $C_2 > 0$. Suppose that $u_0 = \int V\,d\nu > 0$ and

$$T = \int_{u_0}^{\infty} \frac{du}{uG(u)} < +\infty.$$

Then (1) has no probability solution $\mu = (\mu_t)_{t\in[0,T]}$ on $[0,\tau]$ for $\tau \ge T$ with

$$\sup_{t\in[0,T]} \int V(x)\,d\mu_t < \infty.$$

Uniqueness

Obviously, uniqueness is a difficult question even in the linear case (especially with matrix A which is not strictly positive definite), cf.[**?**].

We want to deal with unbounded coefficients and possibly degenerate matrix A. Suppose $A(x,t,\mu) = A(x,t)$.

One possible approach is approximative Holmgren method.

The main idea can be show in a simple case.

Suppose all coefficients are smooth with bounded derivatives and for some increasing G

$$|b(x,t,\mu) - b(x,t,\sigma)| \le G(\|m_t - \sigma_t\|), \tag{3}$$

where $\|\cdot\|$ is Kantorovich-Rubinshtein (Vasserstein) norm.

Suppose we have 2 solutions μ and σ. For any $\psi \in C_0^\infty(\mathbb{R}^d)$ with $|\psi(x)| \le 1$ and $|\nabla\psi(x)| \le 1$ consider an adjoint problem

$$\partial_t f = a^{ij}(x,t)\partial_{x_i x_j} f + b^i(x,t,\sigma)\partial_{x_i} f = 0, \quad f|_{s=t} = \psi.$$

Then f is a smooth function with a gradient, bounded uniformly in ψ. We get

$$\int_{\mathbb{R}^d} \psi(x)\,d\sigma_t = \int_{\mathbb{R}^d} f(x,t)\,d\nu$$

and

$$\int_{\mathbb{R}^d} \psi(x)d\mu_t = \int_{\mathbb{R}^d} f(x,t)\,d\nu + \int_0^t \int_{\mathbb{R}^d} (b(x,s,\mu) - b(x,s,\sigma), \nabla_x f(x,s))\,d\mu_s ds.$$

This yields

$$\int_{\mathbb{R}^d} \psi(x)\,d(\mu_t - \sigma_t) \le C\int_0^t G(\|\mu_s - \sigma_s\|)\,ds,$$

so if G is Osgood (i.e. $\int_0 G^{-1}(u)du = +\infty$) we come to a contradiction.

This method can be extended to the general case and requires one-side Lipschitz condition of b in x, existence of a particular Lyapunov function and (3).

These assumptions, together with some technical regularity assumptions, are sufficient for uniqueness of probability solutions of (1) even with a degenerate matrix A.

The Important Example satisfies all these assumptions.

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