III Kac's process

James Norris

University of Cambridge

Details at arXiv:1405.2742

Kac's process

Write $\mathcal{S}$ for the set of probability measures μ on $\mathbb{R}^3$ such that

$$\langle v, \mu\rangle = \int_{\mathbb{R}^3} v\mu(dv) = 0, \quad \langle |v|^2, \mu\rangle = \int_{\mathbb{R}^3} |v|^2\mu(dv) = 1.$$

Kac's process

Write $\mathcal{S}$ for the set of probability measures μ on $\mathbb{R}^3$ such that

$$\langle v, \mu \rangle = \int_{\mathbb{R}^3} v\mu(dv) = 0, \quad \langle |v|^2, \mu \rangle = \int_{\mathbb{R}^3} |v|^2 \mu(dv) = 1.$$

Write $\mathcal{S}_N$ for the subset of $\mathcal{S}$ consisting of N-particle normalized empirical measures

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \delta_{v_i}.$$

Kac's process

Write $\mathcal{S}$ for the set of probability measures μ on $\mathbb{R}^3$ such that

$$\langle v, \mu \rangle = \int_{\mathbb{R}^3} v\mu(dv) = 0, \quad \langle |v|^2, \mu \rangle = \int_{\mathbb{R}^3} |v|^2 \mu(dv) = 1.$$

Write $\mathcal{S}_N$ for the subset of $\mathcal{S}$ consisting of N-particle normalized empirical measures

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \delta_{v_i}.$$

Given $\mu_0^N \in \mathcal{S}_N$, consider the Markov chain $(\mu_t^N)_{t \geqslant 0}$ in $\mathcal{S}_N$ with the transition rule:

> for every pair of particles v, v_*, at rate $|v - v_*|/N$, draw a sphere with poles v, v_*, choose randomly a new axis for this sphere, with poles v', v_*' say, and replace v, v_* by v', v_*'.

- This is one of a class of processes proposed by Kac in 1954 as models for the evolution by collisions of particle velocities in a spatially homogeneous dilute gas.

- This is one of a class of processes proposed by Kac in 1954 as models for the evolution by collisions of particle velocities in a spatially homogeneous dilute gas.

- The rate of collisions is naturally proportional to the relative speed $|v - v_*|$. Our time-scale normalizes this rate.

- This is one of a class of processes proposed by Kac in 1954 as models for the evolution by collisions of particle velocities in a spatially homogeneous dilute gas.

- The rate of collisions is naturally proportional to the relative speed $|v - v_*|$. Our time-scale normalizes this rate.

- Note that $v' + v'_* = v + v_*$ and write $v' - v'_* = \sigma|v - v_*|$.

- This is one of a class of processes proposed by Kac in 1954 as models for the evolution by collisions of particle velocities in a spatially homogeneous dilute gas.

- The rate of collisions is naturally proportional to the relative speed $|v - v_*|$. Our time-scale normalizes this rate.

- Note that $v' + v'_* = v + v_*$ and write $v' - v'_* = \sigma|v - v_*|$.

- The transition $(v, v_*) \to (v', v'_*)$ models an elastic collision

$$|v|^2 + |v_*|^2 = \frac{1}{2}(|v + v_*|^2 + |v - v_*|^2) = |v'|^2 + |v'_*|^2.$$

- This is one of a class of processes proposed by Kac in 1954 as models for the evolution by collisions of particle velocities in a spatially homogeneous dilute gas.
- The rate of collisions is naturally proportional to the relative speed $|v - v_*|$. Our time-scale normalizes this rate.
- Note that $v' + v'_* = v + v_*$ and write $v' - v'_* = \sigma|v - v_*|$.
- The transition $(v, v_*) \to (v', v'_*)$ models an elastic collision

$$|v|^2 + |v_*|^2 = \frac{1}{2}(|v + v_*|^2 + |v - v_*|^2) = |v'|^2 + |v'_*|^2.$$

- The special case we study, where the direction of $v' - v'_*$ is taken to be uniformly random, corresponds (in 3 dimensions) to a model for *hard sphere* particles – by an elementary geometric calculation.

Kac's purpose was to shed light on the Boltzmann equation which, at least formally, should govern the behaviour of his process in the limit $N \to \infty$.

Indeed he hoped to understand the N-particle system well enough to deduce key properties of the Boltzmann equation.

Kac's purpose was to shed light on the Boltzmann equation which, at least formally, should govern the behaviour of his process in the limit $N \to \infty$.

Indeed he hoped to understand the N-particle system well enough to deduce key properties of the Boltzmann equation.

McKean (1966) and Tanaka (1978, 1983) proved results on other cases of Kac's model.

Sznitman (1984) proved weak convergence in probability for hard spheres to solutions of Boltzmann's equation – formulated as convergence in distribution and asymptotic independence of particles.

Mischler and Mouhot (2013) have established quantitative versions of Sznitman's result (and much more) with good long-time properties.

Kac's purpose was to shed light on the Boltzmann equation which, at least formally, should govern the behaviour of his process in the limit $N \to \infty$.

Indeed he hoped to understand the N-particle system well enough to deduce key properties of the Boltzmann equation.

McKean (1966) and Tanaka (1978, 1983) proved results on other cases of Kac's model.

Sznitman (1984) proved weak convergence in probability for hard spheres to solutions of Boltzmann's equation – formulated as convergence in distribution and asymptotic independence of particles.

Mischler and Mouhot (2013) have established quantitative versions of Sznitman's result (and much more) with good long-time properties.

I will describe a new approach to the question of convergence, based on direct use of martingale estimates, which leads to an explicit pathwise estimate in Wasserstein distance.

Martingales of Kac's process

Encode the jumps in an integer-valued random measure m on $E = \mathbb{R}^3 \times \mathbb{R}^3 \times S^2 \times (0, \infty)$ with atoms at all the quadruples (V, V_*, Σ, T) such that the pair (V, V_*) collides at time T with post-collision velocities (V', V_*') given by $V' - V_*' = \Sigma|V - V_*|$.

Martingales of Kac's process

Encode the jumps in an integer-valued random measure m on $E = \mathbb{R}^3 \times \mathbb{R}^3 \times S^2 \times (0, \infty)$ with atoms at all the quadruples (V, V_*, Σ, T) such that the pair (V, V_*) collides at time T with post-collision velocities (V', V_*') given by $V' - V_*' = \Sigma|V - V_*|$.

The compensator $\bar{m}$ of m is then given by

$$\bar{m}(dv, dv_*, d\sigma, dt) = N|v - v_*|\mu^N_{t-}(dv)\mu^N_{t-}(dv_*)d\sigma dt$$

where $d\sigma$ is the uniform distribution on S^2.

Martingales of Kac's process

Encode the jumps in an integer-valued random measure m on $E = \mathbb{R}^3 \times \mathbb{R}^3 \times S^2 \times (0, \infty)$ with atoms at all the quadruples (V, V_*, Σ, T) such that the pair (V, V_*) collides at time T with post-collision velocities (V', V'_*) given by $V' - V'_* = \Sigma|V - V_*|$.

The compensator $\bar{m}$ of m is then given by

$$\bar{m}(dv, dv_*, d\sigma, dt) = N|v - v_*|\mu^N_{t-}(dv)\mu^N_{t-}(dv_*)d\sigma dt$$

where $d\sigma$ is the uniform distribution on S^2.

Define a random measure M on $(0, \infty) \times \mathbb{R}^3$ by specifying for bounded measurable functions f

$$\begin{aligned} M^f_t &= \int_{(0,t]\times\mathbb{R}^3} f(v)M(ds, dv) \\ &= \int_E \frac{\{f(v') + f(v'_*) - f(v) - f(v_*)\}}{N} 1_{(0,t]}(s)d(m - \bar{m}) \end{aligned}$$

The process $(M_t^f)_{t\geqslant 0}$ is a martingale, with

$$\mathbb{E}|M_t^f|^2 \leqslant 32\|f\|_\infty^2 t/N.$$

Moreover

$$\langle f, \mu_t^N\rangle = \langle f, \mu_0^N\rangle + M_t^f + \int_0^t \langle f, Q(\mu_s^N, \mu_s^N)\rangle ds$$

where, for measures μ, ν on $\mathbb{R}^3$, $Q(\mu, \nu)$ is the signed measure on $\mathbb{R}^3$ given by

$$\langle f, Q(\mu, \nu)\rangle = \int \{f(v')+f(v'_*)-f(v)-f(v_*)\}|v-v_*|\mu(dv)\nu(dv_*)d\sigma.$$

Here $v' = v'(v, v_*, \sigma)$ and $v'_* = v'_*(v, v_*, \sigma)$ are given as always by

$$v' + v'_* = v + v_*, \quad v' - v'_* = \sigma|v - v_*|.$$

Boltzmann's equation

A process $(\mu_t)_{t\geqslant 0}$ in $\mathcal{S}$ is a (measure) solution to the spatially homogeneous Boltzmann equation if, for all bounded measurable functions f of compact support in $\mathbb{R}^3$ and all $t \geqslant 0$

$$\langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \langle f, Q(\mu_s, \mu_s)\rangle ds.$$

Boltzmann's equation

A process $(\mu_t)_{t\geqslant 0}$ in $\mathcal{S}$ is a (measure) solution to the spatially homogeneous Boltzmann equation if, for all bounded measurable functions f of compact support in $\mathbb{R}^3$ and all $t \geqslant 0$

$$\langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \langle f, Q(\mu_s, \mu_s)\rangle ds.$$

Lu and Mouhot (2012) have shown that, for all $\mu_0 \in \mathcal{S}$ there is a unique solution $(\mu_t)_{t\geqslant 0}$ starting from μ_0.

Boltzmann's equation

A process $(\mu_t)_{t\geqslant 0}$ in $\mathcal{S}$ is a (measure) solution to the spatially homogeneous Boltzmann equation if, for all bounded measurable functions f of compact support in $\mathbb{R}^3$ and all $t \geqslant 0$

$$\langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \langle f, Q(\mu_s, \mu_s)\rangle ds.$$

Lu and Mouhot (2012) have shown that, for all $\mu_0 \in \mathcal{S}$ there is a unique solution $(\mu_t)_{t\geqslant 0}$ starting from μ_0.

Remember that $\mathbb{E}|M_t^f|^2 \leqslant 32\|f\|_\infty^2 t/N$ and compare with

$$\langle f, \mu_t^N\rangle = \langle f, \mu_0^N\rangle + M_t^f + \int_0^t \langle f, Q(\mu_s^N, \mu_s^N)\rangle ds.$$

Can we use these equations to see that in the limit $N \to \infty$

$$\mu_0^N \to \mu_0 \quad \text{implies} \quad \mu_t^N \to \mu_t \quad \text{for all } t \geqslant 0?$$

Wasserstein distance

For functions f on $\mathbb{R}^3$ we will write $\|f\|$ for the smallest constant such that, for all v, v',

$$|\hat{f}(v)| \leqslant \|f\|, \quad |\hat{f}(v) - \hat{f}(v')| \leqslant \|f\| |v - v'|.$$

where $\hat{f}(v) = f(v)/(1 + |v|^2)$.

Wasserstein distance

For functions f on $\mathbb{R}^3$ we will write $\|f\|$ for the smallest constant such that, for all v, v',

$$|\hat{f}(v)| \leqslant \|f\|, \quad |\hat{f}(v) - \hat{f}(v')| \leqslant \|f\||v - v'|.$$

where $\hat{f}(v) = f(v)/(1 + |v|^2)$.
We will use on $\mathcal{S}$ the distance function

$$W(\mu, \nu) = \sup_{\|f\|=1} \langle f, \mu - \nu \rangle.$$

Theorem

Assume that $\langle |v|^p, \mu_0 \rangle < \infty$ *for some* $p \geqslant 5$*. Set*

$$\alpha(p) = 1/(6 + 1/(p-3)).$$

For all $\varepsilon > 0$ *and all* $T < \infty$*, there is a constant* $C < \infty$ *such that, for all* $N \in \mathbb{N}$ *and any Kac process* $(\mu_t^N)_{t \geqslant 0}$ *in* $\mathcal{S}_N$*, with probability exceeding* $1 - \varepsilon$*, for all* $t \in [0, T]$*,*

$$W(\mu_t^N, \mu_t) \leqslant C(W(\mu_0^N, \mu_0) + N^{-\alpha(p)}).$$

The constant C *may be chosen to depend only on* ε, λ, p *and* T*, where* λ *is an upper bound for* $\langle |v|^p, \mu_0 \rangle$ *and* $\langle |v|^p, \mu_0^N \rangle$*.*

Theorem

Assume that $\langle |v|^p, \mu_0 \rangle < \infty$ *for some* $p \geqslant 5$. *Set*

$$\alpha(p) = 1/(6 + 1/(p-3)).$$

For all $\varepsilon > 0$ *and all* $T < \infty$, *there is a constant* $C < \infty$ *such that, for all* $N \in \mathbb{N}$ *and any Kac process* $(\mu_t^N)_{t \geqslant 0}$ *in* $\mathcal{S}_N$, *with probability exceeding* $1 - \varepsilon$, *for all* $t \in [0, T]$,

$$W(\mu_t^N, \mu_t) \leqslant C(W(\mu_0^N, \mu_0) + N^{-\alpha(p)}).$$

The constant C *may be chosen to depend only on* ε, λ, p *and* T, *where* λ *is an upper bound for* $\langle |v|^p, \mu_0 \rangle$ *and* $\langle |v|^p, \mu_0^N \rangle$.

A similar estimate, with a different formula for $\alpha(p) > 0$ holds also for $p \in (3, 5)$.

Moment estimates

It is known that for all $p \in \mathbb{N}$ there is a constant C_p depending only on p such that, for all $t \geqslant 0$,

$$\langle |v|^p, \mu_t \rangle \leqslant C_p \langle 1 + |v|^p, \mu_0 \rangle.$$

Moment estimates

It is known that for all $p \in \mathbb{N}$ there is a constant C_p depending only on p such that, for all $t \geqslant 0$,

$$\langle |v|^p, \mu_t \rangle \leqslant C_p \langle 1 + |v|^p, \mu_0 \rangle.$$

Indeed C_p may be chosen so that, for all $t \geqslant 0$ and all $N \in \mathbb{N}$,

$$\mathbb{E}\left(\sup_{s \leqslant t} \langle |v|^p, \mu_t^N \rangle \right) \leqslant C_p (1 + t) \langle 1 + |v|^p, \mu_0^N \rangle.$$

Moment estimates

It is known that for all $p \in \mathbb{N}$ there is a constant C_p depending only on p such that, for all $t \geqslant 0$,

$$\langle |v|^p, \mu_t \rangle \leqslant C_p \langle 1 + |v|^p, \mu_0 \rangle.$$

Indeed C_p may be chosen so that, for all $t \geqslant 0$ and all $N \in \mathbb{N}$,

$$\mathbb{E}\left(\sup_{s \leqslant t} \langle |v|^p, \mu_t^N \rangle \right) \leqslant C_p (1+t) \langle 1 + |v|^p, \mu_0^N \rangle.$$

We will obtain some estimates in terms of the random variables

$$m_p(t) = \sup_{s \leqslant t} \langle 1 + |v|^p, \mu_s^N + \mu_s \rangle.$$

Towards a stability argument

Subtract the Boltzmann equation from the martingale decomposition for Kac's process to obtain, for suitable functions f on $\mathbb{R}^3$ and $\rho_t = \mu_t^N + \mu_t$,

$$\langle f, \mu_t^N - \mu_t \rangle = \langle f, \mu_0^N - \mu_0 \rangle + M_t^f + \int_0^t \langle f, Q(\rho_s, \mu_s^N - \mu_s) \rangle ds.$$

We will treat this as a driven linear equation for $\mu_t^N - \mu_t$ and attempt to show stability around the undriven case.

Towards a stability argument

Subtract the Boltzmann equation from the martingale decomposition for Kac's process to obtain, for suitable functions f on $\mathbb{R}^3$ and $\rho_t = \mu_t^N + \mu_t$,

$$\langle f, \mu_t^N - \mu_t \rangle = \langle f, \mu_0^N - \mu_0 \rangle + M_t^f + \int_0^t \langle f, Q(\rho_s, \mu_s^N - \mu_s) \rangle ds.$$

We will treat this as a driven linear equation for $\mu_t^N - \mu_t$ and attempt to show stability around the undriven case.

Two immediate obstacles

- the process is infinite-dimensional (measure-valued),
- the Boltzmann operator $Q(\rho_t, \cdot)$ is unbounded.

Towards a stability argument

Subtract the Boltzmann equation from the martingale decomposition for Kac's process to obtain, for suitable functions f on $\mathbb{R}^3$ and $\rho_t = \mu_t^N + \mu_t$,

$$\langle f, \mu_t^N - \mu_t \rangle = \langle f, \mu_0^N - \mu_0 \rangle + M_t^f + \int_0^t \langle f, Q(\rho_s, \mu_s^N - \mu_s) \rangle ds.$$

We will treat this as a driven linear equation for $\mu_t^N - \mu_t$ and attempt to show stability around the undriven case.

Two immediate obstacles

- the process is infinite-dimensional (measure-valued),
- the Boltzmann operator $Q(\rho_t, \cdot)$ is unbounded.

So

- we use a finite-dimensional approximation,
- we solve the linear equation explicitly, taking advantage of good parts of the operator.

Branching process

Condition on Kac's process and introduce for $s \geqslant 0$ and $v \in \mathbb{R}^3$ an auxiliary branching process of positive and negative particles starting from a single positive particle at v at time s.

The branching rule is that each positive particle v, at rate

$$|v - v_*|(\mu_t^N + \mu_t)(dv_*)d\sigma dt,$$

dies and is replaced by two positive particles $v' = v'(v, v_*, \sigma)$ and $v'_* = v'_*(v, v_*, \sigma)$ and one negative particle v_*, and a similar rule holds for negative particles.

Branching process

Condition on Kac's process and introduce for $s \geqslant 0$ and $v \in \mathbb{R}^3$ an auxiliary branching process of positive and negative particles starting from a single positive particle at v at time s.

The branching rule is that each positive particle v, at rate

$$|v - v_*|(\mu_t^N + \mu_t)(dv_*)d\sigma dt,$$

dies and is replaced by two positive particles $v' = v'(v, v_*, \sigma)$ and $v'_* = v'_*(v, v_*, \sigma)$ and one negative particle v_*, and a similar rule holds for negative particles.

Write $\Lambda_t^\pm$ for the un-normalized empirical measures of $\pm$ particles at time t. Fix $t \geqslant 0$ and a function f_t on $\mathbb{R}^3$. Define for $s \in [0, t]$

$$f_s(v) = E_{(s,v)}\langle f_t, \Lambda_t^+ - \Lambda_t^- \rangle.$$

Branching process

Condition on Kac's process and introduce for $s \geqslant 0$ and $v \in \mathbb{R}^3$ an auxiliary branching process of positive and negative particles starting from a single positive particle at v at time s.

The branching rule is that each positive particle v, at rate

$$|v - v_*|(\mu_t^N + \mu_t)(dv_*)d\sigma dt,$$

dies and is replaced by two positive particles $v' = v'(v, v_*, \sigma)$ and $v'_* = v'_*(v, v_*, \sigma)$ and one negative particle v_*, and a similar rule holds for negative particles.

Write $\Lambda_t^\pm$ for the un-normalized empirical measures of $\pm$ particles at time t. Fix $t \geqslant 0$ and a function f_t on $\mathbb{R}^3$. Define for $s \in [0, t]$

$$f_s(v) = E_{(s,v)}\langle f_t, \Lambda_t^+ - \Lambda_t^- \rangle.$$

Then

$$\langle f_t, \mu_t^N - \mu_t \rangle = \langle f_0, \mu_0^N - \mu_0 \rangle + \int_{(0,t]\times\mathbb{R}^3} f_s(v) M(ds, dv).$$

Outline of the proof

We assume that $\|f_t\| \leqslant 1$ and use the equation

$$\langle f_t, \mu_t^N - \mu_t \rangle = \langle f_0, \mu_0^N - \mu_0 \rangle + \int_{(0,t]\times\mathbb{R}^3} f_s(v) M(ds, dv).$$

to find a set Ω_0 of high probability and a small bound δ, independent of $t \in [0, T]$ and f_t, such that, on Ω_0,

$$\langle f_t, \mu_t^N - \mu_t \rangle \leqslant \delta.$$

Then, on Ω_0,

$$\sup_{t\leqslant T} W(\mu_t^N, \mu_t) \leqslant \delta.$$

Lemma

For all functions f_t on $\mathbb{R}^3$, the function

$$f_s(v) = E_{(s,v)}\langle f_t, \Lambda_t^+ - \Lambda_t^- \rangle$$

satisfies, for all $s, s' \leqslant t$ and all $v \in \mathbb{R}^3$,

$$\|f_s\| \leqslant C(T)\|f_t\|, \quad |f_s(v) - f_{s'}(v)| \leqslant C(T)(1 + |v|^3)|s - s'|\|f_t\|.$$

Here

$$C(T) = 6(T+1)e^{4Tm_3(T)}, \quad m_3(T) = \sup_{t \leqslant T}\langle 1 + |v|^3, \mu_t + \mu_t^N \rangle.$$

Lemma

For all functions f_t on $\mathbb{R}^3$, the function

$$f_s(v) = E_{(s,v)}\langle f_t, \Lambda_t^+ - \Lambda_t^-\rangle$$

satisfies, for all $s, s' \leqslant t$ and all $v \in \mathbb{R}^3$,

$$\|f_s\| \leqslant C(T)\|f_t\|, \quad |f_s(v) - f_{s'}(v)| \leqslant C(T)(1+|v|^3)|s-s'|\|f_t\|.$$

Here

$$C(T) = 6(T+1)e^{4Tm_3(T)}, \quad m_3(T) = \sup_{t\leqslant T}\langle 1+|v|^3, \mu_t + \mu_t^N\rangle.$$

This allows us to estimate the first term when $\|f_t\| \leqslant 1$

$$\langle f_0, \mu_0^N - \mu_0\rangle \leqslant C(T)W(\mu_0^N, \mu_0), \quad t \leqslant T.$$

We aim to show for the second term an estimate valid with high probability of the form

$$\int_{(0,t]\times\mathbb{R}^3} f_s(v) M(ds, dv) \leqslant CN^{-\alpha}, \quad t \leqslant T.$$

We aim to show for the second term an estimate valid with high probability of the form

$$\int_{(0,t]\times\mathbb{R}^3} f_s(v)M(ds,dv) \leqslant CN^{-\alpha}, \quad t \leqslant T.$$

The integrand $f_s(v)$ depends implicitly on t and is not adapted in the filtration of M.

We seek an estimate uniform in $t \leqslant T$ and $\|f_t\| \leqslant 1$.

Consider the function f on $[0, T] \times \mathbb{R}^3$ given by $f(s, v) = f_{s \wedge t}(v)$.

Cover the set $(0, T] \times (-R, R]^3$ by $n = (T/r) \times (R/r)^3$ disjoint boxes $B_1, \ldots, B_n$, each a translated copy of $(0, r] \times (-r, r]^3$.

Write

$$f = \sum_{k=1}^{n} a_k 1_{B_k} + g$$

where a_k is the average value of f on B_k.

Consider the function f on $[0, T] \times \mathbb{R}^3$ given by $f(s, v) = f_{s \wedge t}(v)$.

Cover the set $(0, T] \times (-R, R]^3$ by $n = (T/r) \times (R/r)^3$ disjoint boxes $B_1, \ldots, B_n$, each a translated copy of $(0, r] \times (-r, r]^3$.

Write

$$f = \sum_{k=1}^{n} a_k 1_{B_k} + g$$

where a_k is the average value of f on B_k.

By the lemma,

$$|a_k| \leqslant 2C(T)(1 + R^2)$$

and, for any $k \geqslant 0$,

$$|g(s, v)| \leqslant 4C(T)(1 + |v|^3)r + 2C(T)(1 + |v|^p)R^{2-p}.$$

Now

$$f = \sum_{k=1}^{n} a_k 1_{B_k} + g$$

so

$$\int_{(0,t]\times\mathbb{R}^3} f_s(v) M(ds, dv) = \sum_{k=1}^{n} a_k M_t^{(k)} + \int_{(0,t]\times\mathbb{R}^3} g(s, v) M(ds, dv)$$

where

$$M_t^{(k)} = \int_0^t \langle 1_{B_k}(s, \cdot), dM_s \rangle.$$

We estimate

$$\sum_{k=1}^{n} a_k M_t^{(k)} \leqslant 2C(T)(1+R^2)\sqrt{nQ_t}$$

where $Q_t = \sum_{k=1}^{n} |M_t^{(k)}|^2$.

Write

$$\Delta_k(s, v, v_*, \sigma) = 1_{B_k}(s, v') + 1_{B_k}(s, v'_*) - 1_{B_k}(s, v) - 1_{B_k}(s, v_*).$$

Then

$$\sum_{k=1}^{n} \Delta_k(s, v, v_*, \sigma)^2$$

$$\leqslant 4 \sum_{k=1}^{n} \{1_{B_k}(s, v') + 1_{B_k}(s, v'_*) + 1_{B_k}(s, v) + 1_{B_k}(s, v_*)\} \leqslant 16.$$

So, by Doob's L^2-inequality,

$$\mathbb{E}\left(\sup_{s \leqslant t} Q_s\right) \leqslant \frac{4}{N} \sum_{k=1}^{n} \int_E \Delta_k(s, v, v_*, \sigma)^2 1_{(0,t]}(s) d\bar{m} \leqslant \frac{128t}{N}.$$

Write

$$\Delta_k(s,v,v_*,\sigma) = 1_{B_k}(s,v') + 1_{B_k}(s,v_*') - 1_{B_k}(s,v) - 1_{B_k}(s,v_*).$$

Then

$$\begin{aligned}
&\sum_{k=1}^{n} \Delta_k(s,v,v_*,\sigma)^2 \\
&\leqslant 4\sum_{k=1}^{n}\{1_{B_k}(s,v') + 1_{B_k}(s,v_*') + 1_{B_k}(s,v) + 1_{B_k}(s,v_*)\} \leqslant 16.
\end{aligned}$$

So, by Doob's L^2-inequality,

$$\mathbb{E}\left(\sup_{s\leqslant t} Q_s\right) \leqslant \frac{4}{N}\sum_{k=1}^{n}\int_E \Delta_k(s,v,v_*,\sigma)^2 1_{(0,t]}(s)\,d\bar{m} \leqslant \frac{128t}{N}.$$

We estimate the final term $\int_{(0,t]\times\mathbb{R}^3} g(s,v)M(ds,dv)$ absolutely.

The main estimate now follows by optimizing over r and R.

Theorem

Assume that $\langle |v|^p, \mu_0 \rangle < \infty$ *for some* $p \geqslant 5$. *Set*

$$\alpha(p) = 1/(6 + 1/(p-3)).$$

For all $\varepsilon > 0$ *and all* $T < \infty$, *there is a constant* $C < \infty$ *such that, for all* $N \in \mathbb{N}$ *and any Kac process* $(\mu_t^N)_{t \geqslant 0}$ *in* $\mathcal{S}_N$, *with probability exceeding* $1 - \varepsilon$, *for all* $t \in [0, T]$,

$$W(\mu_t^N, \mu_t) \leqslant C(W(\mu_0^N, \mu_0) + N^{-\alpha(p)}).$$

The constant C *may be chosen to depend only on* ε, λ, p *and* T, *where* λ *is an upper bound for* $\langle |v|^p, \mu_0 \rangle$ *and* $\langle |v|^p, \mu_0^N \rangle$.