Kinetic and mean-field modeling of financial markets

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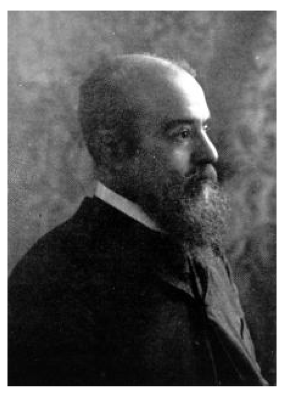
Stylized facts

The financial crisis of recent years showed that many standard financial-market models are unable to explain anomalous events. These anomalous market behaviors called **stylized facts** are universal market properties. **Pareto tails** in wealth distributions and stock returns are well-known examples for such stylized facts.



In 1897 proposed **Vilfredo Pareto** an inverse power law for the wealth or income distribution

$$F_c(w) = \int_w^\infty f(w^\star)\, dw^\star \sim w^{-\mu}.$$

Econophysics

This interdisciplinary field started in the early 90s. **Agent-Based Economic** models are one part of Econophysics. The aim is to observe macroscopic behavior due to microscopic interactions of agents with the help of **Monte Carlo** simulations. One example of such an model is the very influential Levy-Levy-Solomon (LLS) model [1].

The LLS model considers $N \in \mathbb{N}$ financial agents who can invest $\gamma_i,\ i = 1, ..., N$ of their wealth $w_i \in \mathbb{R}_+$ in a stock S and have to invest $1 - \gamma_i$ of their wealth in a safe bond with interest rate $r \in (0, 1)$.

The wealth dynamic of each agent $i = 1, ..., N$ in the time interval $[t-\Delta t, t] \subset [0, \infty),\ \Delta t \in (0, \infty)$ is given by:

$$w_i(t) = (1 - \gamma_i(t-\Delta t))\ w_i(t-\Delta t)\ (1 + r\Delta t) + \gamma_i(t-\Delta t)\ w_i(t-\Delta t)\ (1 + x(S, t, d)\Delta t),$$

where $w_0^i \in \mathbb{R}_+$ and

$$x(S, t, d) := \frac{S(t) - S(t-\Delta t) + \Delta t\ d(t-\Delta t)}{\Delta t\ S(t-\Delta t)},$$

denotes the stock return. The dynamic is driven by the **uncertain dividend stream** $d(\cdot)$, which is modeled as a multiplicative random walk. The stock price is given by the *market clearance condition*

$$S(t) = \frac{1}{n}\sum_{k=1}^N \gamma_k(t)\ w_k(t),$$

where $n \in \mathbb{N}$ is the fixed number of stocks.

Financial agents

Financial agents are often regarded in economics as **rational** agents who aim to maximize their expected utility. This model known as *homo economicus*, has been used frequently in economics and is still in use today.

$$\max_\gamma E[U(w^\gamma)] \approx \max_\gamma \sum_{i=1}^M p_i\ U(w_i^\gamma),$$

where p_i are probabilities and $\sum_{i=1}^M p_i = 1$ has to hold. $U(\cdot)$ is a concave function e.g. the *von Neumann-Morgenstern* utility function:

$$U(w) = \frac{w^{1-\alpha}}{1-\alpha},\ w > 0,\ \alpha \in (1, 2).$$

The seminal paper of Tversky and Kahnemann [2] has shown the limitations of the *homo economicus* approach. The expected value changes to:

$$\max_\gamma E[U(w^\gamma)] \approx \max_\gamma \sum_{i=1}^M W(p_i)\ V(w_i^\gamma).$$

The LLS model considers two different types of agents, regarded as **chartists** and **fundamentalists**. The optimal investment proportion depends in both cases on all parameters of the model.

In general the investment propensity is given by the **algebraic** relation:

$$\gamma_i(t) = g_i\big(S(t), \dot S(t), d(t), r, w_i(t)\big), \quad i = 1, ..., N,$$

where g_i is continuously differentiable and

$$g_i(\cdot) \in (0, 1),$$

holds.

Existing kinetic models

The first realistic kinetic financial-market models have been introduced by Toscani and Pareschi [3]. The key idea is to describe all agents with a probability density

$$f(t, w),\ w \in \mathbb{R}_+.$$

We consider the discrete stochastic process

$$w(t + \Delta t) = (1 - \Delta t\ c)\ w(t) + \Delta t\ m + \Delta t\eta\ w(t),$$

where $m, c > 0$ and η is a normally distributed random variable with zero mean and variance σ^2. Set $\Delta t = 1$, $w' = w(t + \Delta t)$ and $w = w(t)$, we observe:

$$w' = (1 - c)\ w + m + \eta\ w.$$

This linear dynamic can be interpreted as a collision rule in the spirit of kinetic theory. The weak form of the homogeneous linear kinetic equation is given by:

$$\frac{d}{dt}\int_{\mathbb{R}} f(w, t)\ \phi(w)\ dw = \int_{\mathbb{R}_+}\int_{\mathbb{R}} \Psi(\eta)\mathbb{1}_{\{w'>0\}}(w)\ f(w, t)\ (\phi(w') - \phi(w))\ d\eta dw.$$

Here, Ψ is the density of our random variable and the indicator function $\mathbb{1}_{\{w'>0\}}(\cdot)$ is needed in order to preserve the positivity of the post interaction wealth. This equation can be also considered to be a **Master Equation** of our microscopic stochastic process. This approach was first introduced by Toscani and Pareschi [3] and applied to the LLS model by Cordier et al. [4]. In the right asymptotic limit they observe the weak form of

$$\frac{\partial}{\partial t} f(w, t) = \frac{\partial}{\partial w}[(cw - m)\ f(w, t)] + \frac{\lambda}{2}\frac{\partial^2}{\partial w^2}[w^2 f(w, t)].$$

The **steady state** of the Fokker-Planck equation is then given by:

$$g_\infty(w) = \bar n\ \exp\left\{-\frac{\mu - 1}{w}\frac{m}{c}\right\}\ \frac{1}{w^{1+\mu}},$$

where $\mu = 1 + \frac{2c}{\lambda}$ and $\bar n$ is a normalization constant given by

$$\bar n = \frac{\left((\mu-1)\frac{m}{c}\right)^\mu}{\Gamma(\mu)}.$$

Project goals

- Rigorous derivation of the mean-field limit for the LLS model. This could be an **algebraic-differential** microscopic model.

$$\dot w_i(t) = (1 - \gamma_i(t))\ r\ w_i(t) + \gamma_i(t)\ w_i(t)\ \frac{\dot S(t) + d(t)}{S(t)}, \quad w_i(0) = w_0^i,$$

$$S(t) = \frac{1}{n}\sum_{k=1}^N \gamma_k(t)\ w_k(t),$$

$$\gamma_i(t) = g_i\big(S(t), \dot S(t), d(t), r, w_i(t)\big), \quad i = 1, ..., N.$$

  Advantage: Start from the microscopic level!
  Simplifications:

  i) $\gamma_i(t) = \tilde c_i \in (0, 1)$,
  ii) $d(\cdot)$ deterministic.

- Mathematical modeling of financial agents.
  Question:
  What is the right mathematical way to express the investment rule of each agent?
- Analyse the relation between irrational behavior of agents and the appearance of stylized facts.

References

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