Boltzmann type control of opinion consensus through leaders

Università degli Studi di Ferrara - Ex labore fructus -

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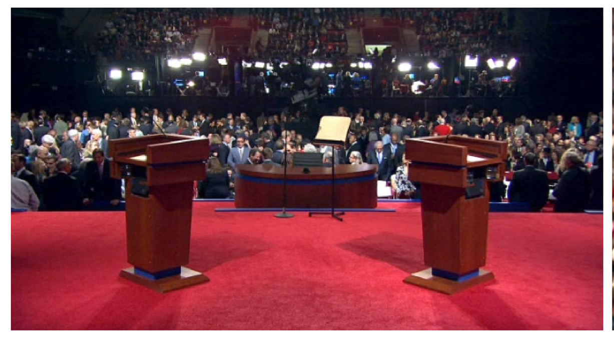
Introduction

We introduce a hierarchical opinion formation dynamics where the leaders aim at controlling the followers through a suitable cost function which characterizes the leaders strategy in trying to influence the followers opinion. Based on this microscopic model, we develop a Boltzmann type optimal control approach following the ideas recently presented in [1].



Microscopic modeling

We assume two populations one of **followers** and one of **leaders** with opinions $w_i, \tilde{w}_k \in I = [-1, 1]$ respectively evolving according to

$$\begin{cases} \dot{w}_i = \dfrac{1}{N_F}\sum_{j=1}^{N_F} P(w_i, w_j)(w_j - w_i) + \dfrac{1}{N_L}\sum_{h=1}^{N_L} S(w_i, \tilde{w}_h)(\tilde{w}_h - w_i), \\ \dot{\tilde{w}}_k = \dfrac{1}{N_L}\sum_{h=1}^{N_L} R(\tilde{w}_k, \tilde{w}_h)(\tilde{w}_h - \tilde{w}_k) + u, \\ w_i(0) = w_{i,0} \qquad \tilde{w}_k(0) = \tilde{w}_{k,0}, \qquad i = 1, ..., N_F \quad k = 1, ..., N_L. \end{cases} \tag{1}$$

- $P, R, S : I \times I \to [0, 1]$ measure the strength of interactions
- u characterizes the **strategy** of the leaders and it is solution of the **optimal control problem**

$$u = \text{argmin}\{J(u, \underline{w}, \underline{\tilde{w}})\},$$

$$J(u, \underline{w}, \underline{\tilde{w}}) = \frac{1}{2}\int_0^T \left\{\frac{\psi}{N_L}\sum_{h=1}^{N_L}(\tilde{w}_h - w_d)^2 + \frac{\mu}{N_L}\sum_{h=1}^{N_L}(\tilde{w}_h - m_F)^2\right\} ds + \int_0^T \frac{\nu}{2}u^2 ds$$

  - $\psi, \mu \geq 0 \qquad \psi + \mu = 1$
  - $w_d \in I$ is the target opinion
  - m_F is the average opinion of the followers group.

Instantaneous binary control

- Split the time interval $[0, T]$ in M time intervals of length Δt and consider $t^n = \Delta t\, n$ for every $n = 1, ..., M$.
- Solve sequentially the **optimal binary control problem** in each time interval

$$\begin{cases} w_i^{n+1} = & w_i^n + \alpha P(w_i^n, w_j^n)(w_j^n - w_i^n) + \alpha S(w_i^n, \tilde{w}_l^n)(\tilde{w}_l^n - w_i^n) \\ w_j^{n+1} = & w_j^n + \alpha P(w_j^n, w_i^n)(w_i^n - w_j^n) + \alpha S(w_j^n, \tilde{w}_l^n)(\tilde{w}_l^n - w_j^n) \end{cases}$$

$$\begin{cases} \tilde{w}_k^{n+1} & = \tilde{w}_k^n + \alpha R(\tilde{w}_k^n, \tilde{w}_h^n)(\tilde{w}_h^n - \tilde{w}_k^n) + 2\alpha u^n \\ \tilde{w}_h^{n+1} & = \tilde{w}_h^n + \alpha R(\tilde{w}_h^n, \tilde{w}_k^n)(\tilde{w}_k^n - \tilde{w}_h^n) + 2\alpha u^n \end{cases}$$

  where $\alpha = \Delta t/2$ and the control variable u^n is solution of

$$\text{argmin}\left\{\frac{\alpha\psi}{2}\sum_{p=\{k,h\}}(\tilde{w}_p^n - w_d)^2 + \frac{\alpha\mu}{2}\sum_{p=\{k,h\}}(\tilde{w}_p^n - m_F^n)^2 + \nu(u^n)^2\right\}.$$

- The **implicit feedback control** u^n can be inverted. We obtain

$$2\alpha u^n = -\sum_{p=\{k,h\}}\frac{\beta}{2}\left[\psi(\tilde{w}_p^n - w_d) + \mu(\tilde{w}_p^n - m_F^n)\right] - \frac{\alpha\beta}{2}(R(\tilde{w}_k^n, \tilde{w}_h^n) - R(\tilde{w}_h^n, \tilde{w}_k^n))(\tilde{w}_h^n - \tilde{w}_k^n),$$

  where $\beta = 4\alpha^2/(\nu + 4\alpha^2)$.

Boltzmann type control

Introduce a density distribution of followers $f_F(w, t)$ and leaders $f_L(\tilde{w}, t)$

$$\int_I f_F(w, t)\, dw = 1, \qquad \int_I f_L(\tilde{w}, t)\, d\tilde{w} = \rho \leq 1.$$

Post-interaction opinions are given by

- **Leader-leader**

$$\begin{cases} \tilde{w}^* = \tilde{w} + \alpha R(\tilde{w}, \tilde{v})(\tilde{v} - \tilde{w}) + 2\alpha u + \tilde{\theta}_1 \tilde{D}(\tilde{w}) \\ \tilde{v}^* = \tilde{v} + \alpha R(\tilde{v}, \tilde{w})(\tilde{w} - \tilde{v}) + 2\alpha u + \tilde{\theta}_2 \tilde{D}(\tilde{v}), \end{cases}$$

  where the feedback control becomes

$$2\alpha u = -\frac{\beta}{2}\left[\psi\left((\tilde{w} - w_d) + (\tilde{v} - w_d)\right) + \mu\left((\tilde{w} - m_F) + (\tilde{v} - m_F)\right)\right] - \frac{\alpha\beta}{2}(R(\tilde{w}, \tilde{v}) - R(\tilde{v}, \tilde{w}))(\tilde{v} - \tilde{w}).$$

- **Follower-follower**

$$\begin{cases} w^* = w + \alpha P(w, v)(v - w) + \theta_1 D(w) \\ v^* = v + \alpha P(v, w)(w - v) + \theta_2 D(v). \end{cases}$$

- **Follower-leader**

$$\begin{cases} w^{**} = w + \alpha S(w, \tilde{v})(\tilde{v} - w) + \hat{\theta}\hat{D}(w) \\ \tilde{v}^{**} = \tilde{v}. \end{cases}$$

Where $\tilde{\theta}_{1,2}, \theta_{1,2}, \hat{\theta}$ are random variables with zero mean and finite variance $\tilde{\sigma}^2, \sigma^2, \hat{\sigma}^2$ and $0 \leq \tilde{D}, D, \hat{D} \leq 1$ represent the local relevance of diffusion.

Boltzmann dynamic

For a test function φ we describe the evolution of $f_F(w, t)$ and $f_L(\tilde{w}, t)$ thanks to the **integro-differential equation of Boltzmann type**

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_I \varphi(w) f_F(w, t) = (Q_F(f_F, f_F), \varphi) + (Q_{FL}(f_L, f_F), \varphi) \\ \dfrac{d}{dt}\displaystyle\int_I \varphi(\tilde{v}) f_L(\tilde{v}, t) d\tilde{v} = (Q_L(f_L, f_L), \varphi), \end{cases} \tag{2}$$

where $Q_F(\cdot,\cdot), Q_{FL}(\cdot,\cdot), Q_L^C(\cdot,\cdot)$ are Boltzmann collisional operators

$$(Q_F(f_F, f_F), \varphi) = \eta_F \left\langle \int_{I^2} (\varphi(w^*) - \varphi(w)) f_F(w, t) f_F(v, t) dw dv \right\rangle$$

$$(Q_{FL}(f_F, f_L), \varphi) = \eta_{FL} \left\langle \int_{I^2} (\varphi(w^{**}) - \varphi(w)) f_F(w, t) f_L(\tilde{v}, t) dw d\tilde{v} \right\rangle$$

$$(Q_L(f_L, f_L), \varphi) = \eta_L \left\langle \int_{I^2} (\varphi(\tilde{w}^*) - \varphi(\tilde{w})) f_L(\tilde{w}, t) f_L(\tilde{v}, t) d\tilde{w} d\tilde{v} \right\rangle.$$

Under the simplified hypothesis $P \equiv S \equiv 1$ and $R(\tilde{w}, \tilde{v}) = R(\tilde{v}, \tilde{w})$ we prove that in absence of diffusion, since $m_F, m_L \to w_d$ as $t \to \infty$, the quantities

$$\int_I f_F(w, t)(w - w_d)^2 dw = E_F(t) + w_d^2 - 2m_F(t) w_d$$

$$\frac{1}{\rho}\int_I f_L(\tilde{w}, t)(\tilde{w} - w_d)^2 d\tilde{w} = E_L(t) + w_d^2 - 2m_L(t) w_d$$

converge to zero as soon as $t \to \infty$, i.e. the steady state solutions are **Dirac delta functions** centered in the target opinion w_d.

Quasi invariant opinion limit

Let consider the scaling parameter $\varepsilon > 0$ then

$$\alpha = \varepsilon, \qquad \nu = \varepsilon\kappa, \qquad \sigma^2 = \varepsilon\varsigma^2, \qquad \hat{\sigma}^2 = \varepsilon\hat{\varsigma}^2, \qquad \tilde{\sigma}^2 = \varepsilon\tilde{\varsigma}^2,$$
$$\eta_F = \frac{1}{c_F\varepsilon}, \qquad \eta_{FL} = \frac{1}{c_{FL}\varepsilon}, \qquad \eta_L = \frac{1}{c_L\varepsilon}, \qquad \beta = \frac{4\varepsilon}{\kappa + 4\varepsilon}. \tag{3}$$

This approach leads to a constrained Fokker-Plank system for the description of the opinion distribution of leaders and followers.

Fokker-Plank equations

Thanks to the scaling (3) for the Boltzmann equations (2), up to a second order Taylor expansion, we obtain as long as $\varepsilon \to 0$

- Fokker-Plank equation for the followers distribution f_F

$$\frac{\partial f_F}{\partial t} + \frac{\partial}{\partial w}\left(\frac{1}{c_F}K_F[f_F](w) + \frac{1}{c_{FL}}K_{FL}[f_L](w)\right) f_F(w) = \frac{1}{2}\frac{\partial^2}{\partial \tilde{w}^2}\left(\frac{\varsigma^2}{c_F}D^2(w) + \frac{\hat{\varsigma}^2\rho}{c_{FL}}\hat{D}^2(w)\right) f_F(w),$$

$$K_F[f_F](w) = \int_I P(w, v)(v - w) f_F(v, t) dv, \qquad K_{FL}[f_L](w) = \int_I S(w, \tilde{w})(\tilde{w} - w) f_L(\tilde{w}) d\tilde{w}.$$

- Fokker-Plank equation for the leaders distribution f_L

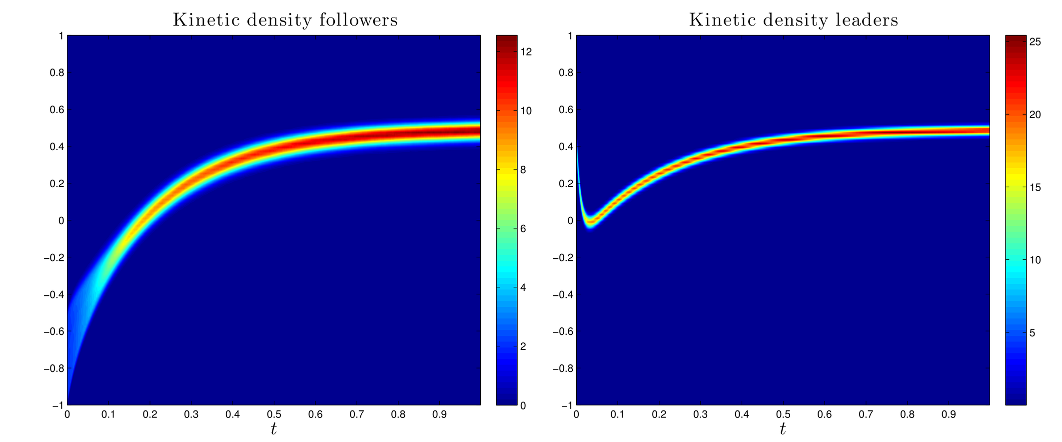
$$\frac{\partial f_L}{\partial t} + \frac{\partial}{\partial \tilde{w}}\left(\frac{\rho}{c_L}H[f_L](\tilde{w}) + \frac{1}{c_L}K_L[f_L](\tilde{w})\right) f_L(\tilde{w}) = \frac{1}{2}\frac{\tilde{\varsigma}^2\rho}{c_L}\frac{\partial^2}{\partial \tilde{w}^2}\tilde{D}^2(\tilde{w}) f_L(\tilde{w}),$$

$$K[f_L](\tilde{w}) = \int_I R(\tilde{w}, \tilde{v})(\tilde{v} - \tilde{w}) f_L(\tilde{v}, t) d\tilde{v}, \qquad H[f_L](\tilde{w}) = \frac{2\psi}{\kappa}(\tilde{w} + m_L(t) - 2w_d) + \frac{2\mu}{\kappa}(\tilde{w} + m_L(t) - 2m_F(t)).$$

Numerical simulations

Numerical result demonstrate the validity of the Boltzmann type control approach and the capability of the leaders control to strategically lead the followers opinion. We assume that the five per cent of the population is composed by opinion leaders and in every figure the leaders profiles have been magnified by a factor ten. In all the results we chose as scaling parameter $\varepsilon = 0.01$ and we use the notation $\hat{c}_{FL} = c_{FL}/\rho$ and $\hat{c}_L = c_L/\rho$.

Test 1: Leaders driving followers



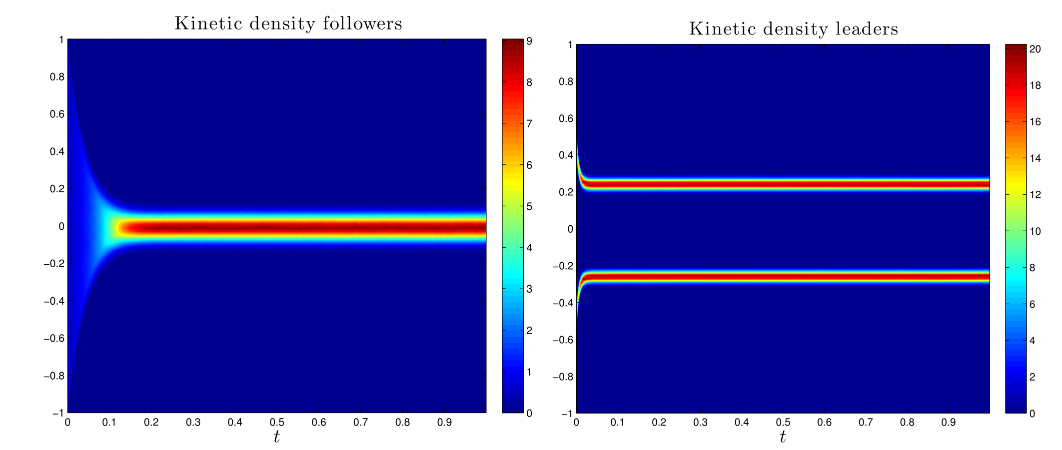
- Simulation of the model (2) for initial distributions $f_F \sim U([-1, -0.5])$ and $f_L \sim N(w_d, 0.05)$, $w_d = 0.5$,
  - $\hat{c}_{FL} = 0.1, \hat{c}_L = 0.1, c_F = 1$
  - $\psi = \mu = 0.5$.

Test 2: Multiple leaders population

Let $M > 0$ be the number of families of leaders such that

$$\int_I f_{L_p}(\tilde{w}) d\tilde{w} = \rho_p \qquad \hat{c}_{FL_p} = \frac{c_{FL_p}}{\rho_p} \qquad p = 1, ..., M.$$

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_I \varphi(w) f_F(w, t) dw = (Q_F(f_F, f_F), \varphi) + \sum_{p=1}^{M}(Q_{FL}(f_{L_p}, f_F), \varphi) \\ \dfrac{d}{dt}\displaystyle\int_I \varphi(\tilde{w}) f_{L_p}(\tilde{w}, t) d\tilde{w} = (Q_L(f_{L_p}, f_{L_p}), \varphi), \quad p = 1, \ldots, M. \end{cases} \tag{4}$$



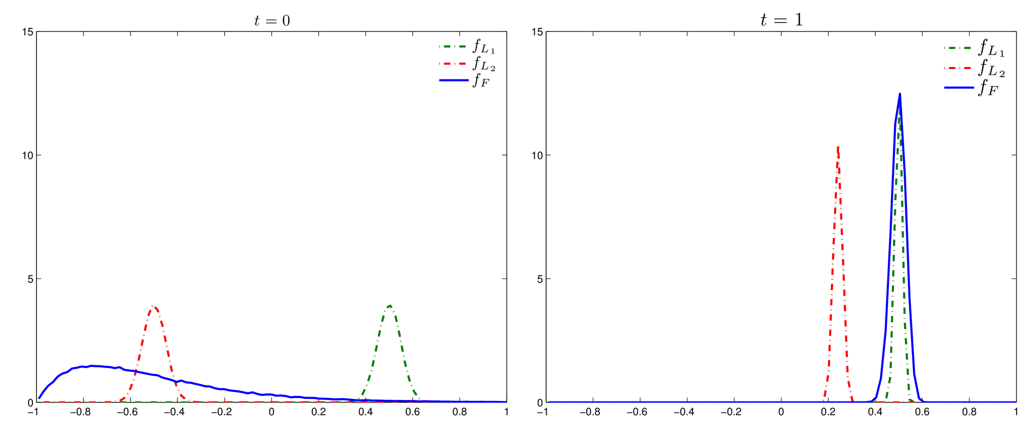
- Simulation of the model (4) for initial distributions $f_F \sim U([-1, 1])$ and $f_{L_p} \sim N(w_{d_p}, 0.05)$, $w_{d_1} = -w_{d_2} = 0.5$
  - $\hat{c}_{FL_1} = \hat{c}_{FL_2} = 0.1, \hat{c}_{L_1} = \hat{c}_{L_2} = 0.1, c_F = 1,$
  - $\psi_1 = \psi_2 = 0.5$ and $\mu_1 = \mu_2 = 0.5$.

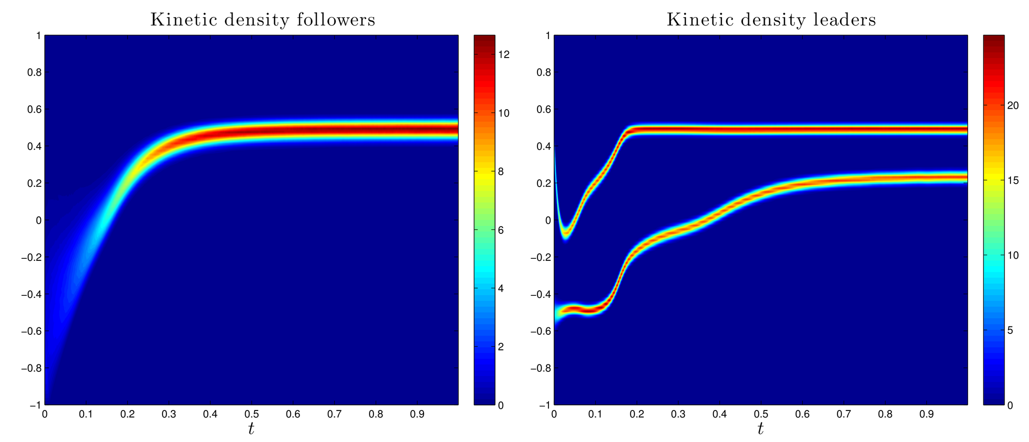
Test 3: Time-dependent strategies

Let us define for every $t \in [0, T]$

$$\psi_p(t) = \frac{1}{2}\int_{w_{d_p} - \delta}^{w_{d_p} + \delta} f_F(w) dw + \frac{1}{2}\int_{m_{L_p} - \tilde{\delta}}^{m_{L_p} + \tilde{\delta}} f_F(w) dw, \qquad \mu_p(t) = 1 - \psi_p(t)$$

- $\delta, \tilde{\delta} \in [0, 1]$ fixed
- m_{L_p} average opinion of the p-th leader.





- Simulation of the model (4) with time dependent coefficients,
  - skewed followers initial distribution $f_F \sim \Gamma(2, \frac{1}{4})$, leaders initial distributions $f_{L_1} \sim N(w_{d_1}, 0.05)$ and $f_{L_2} \sim N(w_{d_2}, 0.05)$,
  - $\hat{c}_{FL_1} = 0.1, \hat{c}_{FL_2} = 1, \hat{c}_{L_1} = \hat{c}_{L_2} = 0.1, c_F = 1$.

References

[1] G. Albi, M. Herty, and L. Pareschi. Kinetic description of optimal control problems in consensus modeling. To appear on *Comm. Math. Sci.*, 2014.

[2] G. Albi, L. Pareschi and M. Zanella. Boltzmann type control of opinion consensus through leaders. arxiv.org/abs/1405.0736