# PDEs for neural networks

## analysis, simulations and behaviour

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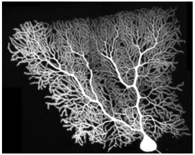
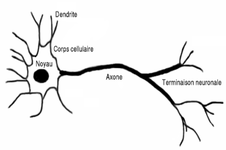
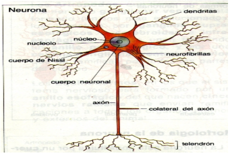
UIMP, august 2018





# Plan of the course

**I.** The single neuron,

- **I. 1.** Excitable systems
- **I. 2.** slow-fast dynamics,
- **I. 3.** Integrate&Fire model, role of noise
- **I. 4.** Distribution of neurons

**II.** Networks, examples

**III.** Networks, Leaky noisy I&F

**IV.** Networks, time elapsed models

# Single neuron

Electrically active cells are described by an **action potential** $V(t)$

Models are well established

- Hodgkin-Huxley
- FitzHugh-Nagumo
- Morris-Lecar
- Mitchell-Schaeffer

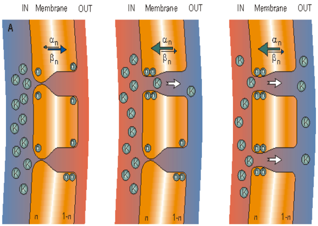
$$
\begin{aligned}
C\frac{dv}{dt} &= I - g_{Na}m^3h(V - V_{Na}) - g_K n^4(V - V_K) - g_L(V - V_L)\\
\frac{dm}{dt} &= a_m(V)(1-m) - b_m(V)m\\
\frac{dh}{dt} &= a_h(V)(1-h) - b_h(V)h\\
\frac{dn}{dt} &= a_n(V)(1-n) - b_n(V)n\\
a_m(V) &= .1(V+40)/(1-\exp(-(V+40)/10))\\
b_m(V) &= 4\exp(-(V+65)/18)\\
a_h(V) &= .07\exp(-(V+65)/20)\\
b_h(V) &= 1/(1+\exp(-(V+35)/10))\\
a_n(V) &= .01(V+55)/(1-\exp(-(V+55)/10))\\
b_n(V) &= .125\exp(-(V+65)/80)
\end{aligned}
$$

# Single neuron

The class of Morris-Lecar is typically

$$\begin{cases} \dfrac{dV(t)}{dt} = \displaystyle\sum_{k=1}^{I} g_k(t)(V_k - V(t)) + I(t), \\ \dfrac{dg_k(t)}{dt} = \dfrac{G_k\big(V(t)\big) - g_k(t)}{\tau_k}, \qquad g_k(0) \geq 0, \quad k = 1, 2, ..., k_M, \end{cases}$$

- The index $k$ refers to ionic channels/conductances along the nerve (Ca, K, Na, Cl...)




From J. Malmivuo and R. Plonsey, Principles and Appl. of bioelectric and biomagnetic fields, OUP 1995

# Single neuron

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- The index $k$ refers to ionic channels/conductances along the nerve (Ca, K, Na, Cl...)
- The $V_k$ are called the "reversal potentials" (Nernst-Planck theory)
- The leak $V_L$ is used to aggregate some of them
- $\tau_k$ can be $\ll 1$
- Sharp nonlinearities $G_k$ (sigmoids)

# Single neuron

The class of Morris-Lecar is typically

$$
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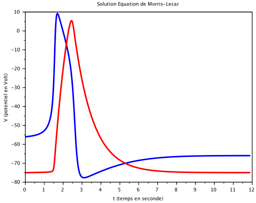
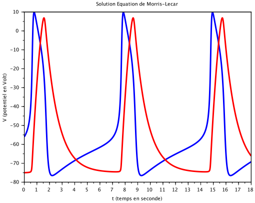
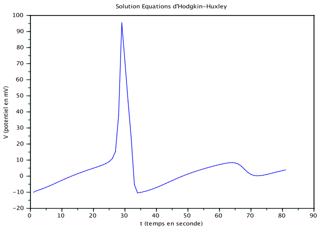
$$

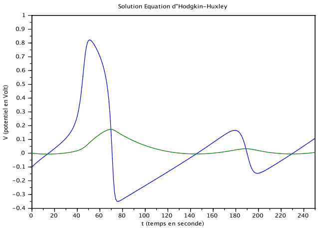



spike $\uparrow$ $\uparrow$Hyperpolarisation

# Single neuron

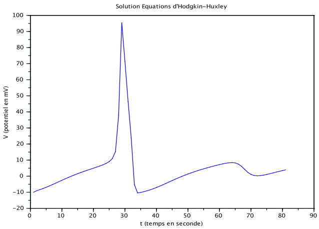


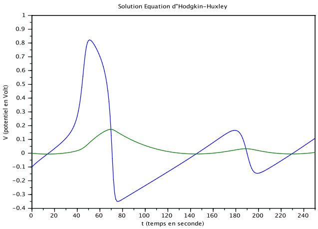




Solutions of Hodgkin-Huxley's model and of FitzHugh-Nagumo's model

- these models are accurate
- represent the property of **excitability** and **hyperpolarization**

# Single neuron

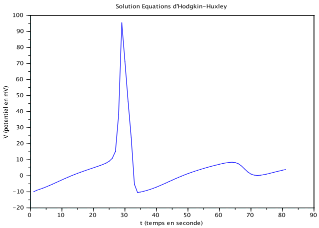


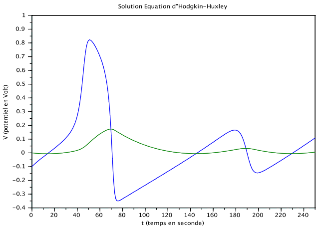




Solutions of Hodgkin-Huxley's model and of FitzHugh-Nagumo's model

- these models are accurate
- represent the property of **excitability**
  - A small perturbation generates a large trajectory
  - Return to equilibrium
  - The trajectory depends very little on the perturbation

# Single neuron




Solutions of Hodgkin-Huxley's model and of FitzHugh-Nagumo's model

- These models are accurate BUT

- difficult to understand why they are **excitable**
- expensive for large assemblies of neurones
- do not explain properties of large assemblies
- This motivates using simpler models

# Single neuron : FHN

FitzHugh-Nagumo

$$\begin{cases} \varepsilon \dot{v}(t) = f(v(t)) - w(t), & v(t=0) = v^0, \\ \dot{w}(t) = v(t) - v^* - \alpha w(t) & w(t=0) = w^0. \end{cases}$$

It can be derived from the Morris-Lecar model

$$\begin{cases} \frac{dV(t)}{dt} = g_L(V_L - V(t)) + G_{Na}(V(t))(V_{Na} - V(t)) + g_K(t)(V_K - V(t)) \\ \\ \dfrac{dg_K(t)}{dt} = \dfrac{G_K\big(V(t)\big) - g_K(t)}{\tau_K} \end{cases}$$

$$V_K < V_L < V_{Na}$$

# Single neuron : FHN

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$$v(t) = \ln(V(t) - V_K)$$

# Single neuron : FHN

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$$\frac{dv(t)}{dt} = \underbrace{\frac{g_L(V_L - V(t)) + G_{Na}(V(t))(V_{Na} - V(t))}{V(t) - V_K}}_{:=F(v(t))} - g_K(t)$$

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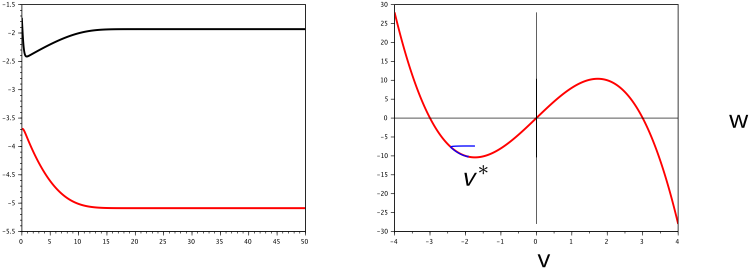


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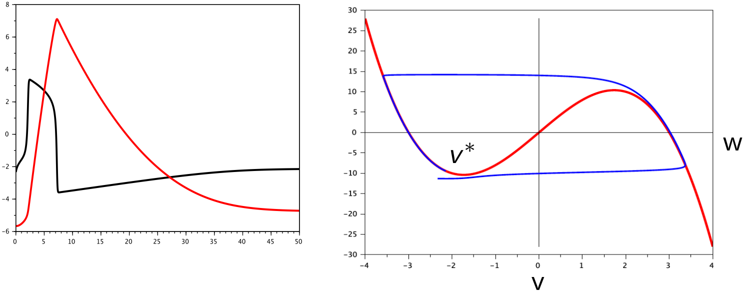


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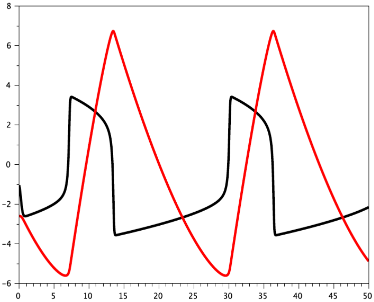


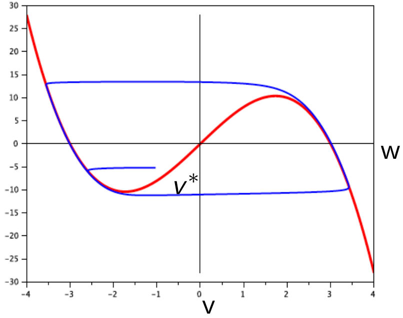


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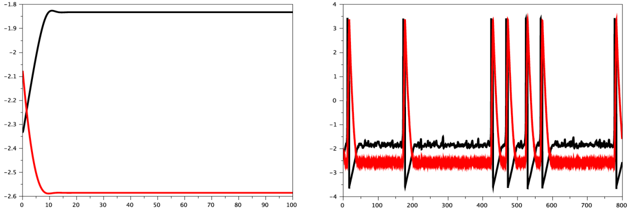
# Single neuron : Role of noise




$$\begin{cases} \dfrac{dv(t)}{dt} = f(v(t)) - w(t), \\ \dfrac{dw(t)}{dt} = v(t) - v^* + \sigma \dfrac{dB(t)}{dt}. \end{cases}$$

# Single neuron : S-F dynamics

Slow-fast dynamics

$$\begin{cases} \varepsilon \dot{v}_\varepsilon(t) = f(v_\varepsilon(t)) - w_\varepsilon(t), & v_\varepsilon(t=0) = v^0, \\ \dot{w}_\varepsilon(t) = v_\varepsilon(t) - v^* & w_\varepsilon(t=0) = w^0. \end{cases}$$

Theorem As $\varepsilon \to 0$, we have

- $v_\varepsilon(t) \to v(t)$ a.e.,
- $w_\varepsilon(t) \to w(t)$ uniformly (locally)

$$\frac{dw(t)}{dt} = Q_\pm(w(t)) - v^*, \qquad v(t) = Q_\pm(w(t)) \Leftrightarrow f(v) = w$$

Slow-fast dynamics

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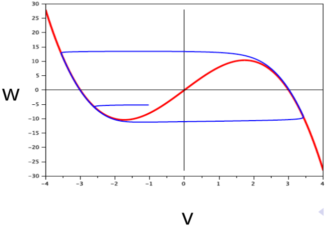

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Proof (1)

$$\begin{aligned} |f(v_\varepsilon(t)) - w_\varepsilon(t)|^2 &= \varepsilon \dot{v}_\varepsilon(t)[f(v_\varepsilon(t)) - w_\varepsilon(t)] \\ &= \varepsilon \tfrac{d}{dt}[F(v_\varepsilon(t)) - v_\varepsilon(t) w_\varepsilon(t)] + \varepsilon v_\varepsilon \tfrac{dw_\varepsilon}{dt} \end{aligned}$$

with $F' = f$

# Single neuron : S-F dynamics

$$\begin{cases} \varepsilon \dot{v}_\varepsilon(t) = f(v_\varepsilon(t)) - w_\varepsilon(t), & v_\varepsilon(t=0) = v^0, \\ \dot{w}_\varepsilon(t) = v_\varepsilon(t) - v^* & w_\varepsilon(t=0) = w^0. \end{cases}$$

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$$\frac{1}{\varepsilon} \int_0^T |f(v_\varepsilon(t)) - w_\varepsilon(t)|^2 dt = F(v_\varepsilon) - v_\varepsilon w_\varepsilon|_0^T + \int_0^T v_\varepsilon(v_\varepsilon - v^*) dt$$

and this is bounded (assuming solutions are bounded).

# Single neuron : S-F dynamics

$$\begin{cases} \varepsilon \dot{v}_\varepsilon(t) = f(v_\varepsilon(t)) - w_\varepsilon(t), & v_\varepsilon(t=0) = v^0, \\ \dot{w}_\varepsilon(t) = v_\varepsilon(t) - v^* & w_\varepsilon(t=0) = w^0. \end{cases}$$

Proof (2)

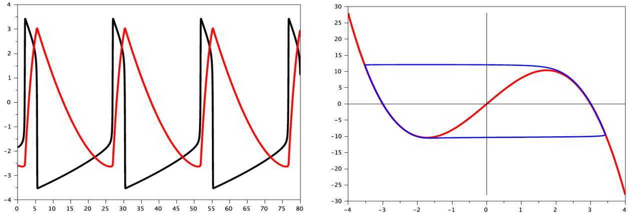
$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^{v_\varepsilon(t)} |f(z) - w_\varepsilon(t)|^2 dz &= \dot{v}_\varepsilon(t)|f(v_\varepsilon(t)) - w_\varepsilon(t)|^2 + Bdd \\ &= \underbrace{\varepsilon \dot{v}_\varepsilon(t)}_{bounded} \underbrace{\frac{|f(v_\varepsilon(t)) - w_\varepsilon(t)|^2}{\varepsilon}}_{step\ 1} + Bdd \end{aligned}$$

Which means that, afer extraction

$$\int_0^{v_\varepsilon(t)} |f(z) - w_\varepsilon(t)|^2 dz$$

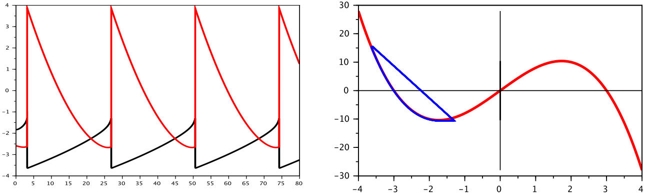
converges a.e.

# Single neuron : I&F



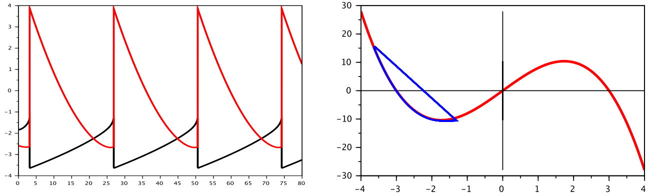

(FitzHugh-Nagumo, fast discharge) solution of a variant of the FitzHugh-Nagumo system

$$\begin{cases} \dfrac{dv(t)}{dt} = h(v(t)) + I(t), & \tau_i \le t < \tau_{i+1}, \\ v(\tau_i^+) = V_R, \qquad \lim_{t\to\tau_{i+1}} v(t) = v(\tau_{i+1}^-) = V_F. \end{cases}$$




Solution of the integrate-and-fire system

# Single neuron : I&F




Solution of the integrate-and-fire system

$$\begin{cases} \frac{dV(t)}{dt} = h(V(t)) + I(t), & \tau_i \leq t < \tau_{i+1}, \\ V(\tau_i^+) = V_R, & \lim_{t \to \tau_{i+1}} V(t) = V(\tau_{i+1}^-) = V_F. \end{cases}$$

With $h \leq 0$

When $I(t) = 0$. $V(t) \to V_R$ (relaxation)

When $I(t) \gg 1$ periodic solutions appear

# Voltage-conductance- Eulerian

Vlasov type equation

$$\frac{\partial}{\partial t}p(v,g,t)+\frac{\partial}{\partial v}\left[\left(g_L(V_L-v)+g(V_E-v)\right)p(v,g,t)\right]$$

$$+\frac{\partial}{\partial g}\left[\frac{G(v,t)-g}{\sigma_E}p(v,g,t)\right]-\frac{a}{\sigma_E}\frac{\partial^2}{\partial g^2}p(v,g,t)=0,$$

$$v\in(V_L,V_E),\ g\geq 0,$$

Boundary conditions :

- Zero flux at $V_L < V_E$
- No flux condition at $g = 0$

Mathematical interest : Sub-elliptic fluxes

# Voltage-conductance- Eulerian

Similar to the Kinetic Fokker-Plack model of interacting particles

$$\frac{\partial}{\partial t} p(x, v, t) + v.\nabla_x p - \mathrm{div}_v(vp) - \Delta_v p = 0$$

Regularizing effects, time decay $M = \exp(-|v|^2/2), \ u = p/M$

$$\frac{1}{2}\frac{d}{dt}\int M|\partial_v u + \partial_x u|^2 \le -\int M|\partial_v u + \partial_x u|^2.$$

Bouchut, Desvillettes, Villani, Hérau

Dolbeault, Mouhot, Schmeiser, Herda, Arnold

Càceres, Carrillo, Goudon

Liu Liu and S. Jin

# I&F- Eulerian

$$\begin{cases} \frac{dv(t)}{dt} = h(v(t)) + I(t), & \tau_i \leq t < \tau_{i+1}, \\ v(\tau_i^+) = V_R, & \lim_{t \to \tau_{i+1}} v(t) = v(\tau_{i+1}^-) = V_F. \end{cases}$$

$$\frac{\partial}{\partial t} p(v,t) + \frac{\partial}{\partial v}[(h(v) + I)p(v,,t) = 0, 0 < V_R < V_F,$$

$$(h(V_R) + I)p(v_R, t) = N(t) := (h(V_F) + I)p(v_F, t)$$

Assuming $(h(V_R) + I) > 0$,

$(h(V_F) + I) > 0$

# Voltage-conductance I&F

$$\frac{\partial}{\partial t}p(v,g,t) + \frac{\partial}{\partial v}\Big[\big(-g_L(V_L-v)+g(V_E-v)\big)p(v,g,t)\Big]$$

$$+\frac{\partial}{\partial g}\Big[\frac{G(t)-g}{\sigma_E}p(v,g,t)\Big] - \frac{a}{\sigma_E}\frac{\partial^2}{\partial g^2}p(v,g,t) = 0,$$

$$v \in (V_L, V_F),\; g \geq 0,$$

Boundary conditions :

- outgoing Flux $N(g,t)$ at $V_F < V_E$ enters at $v = V_L$
- No flux condition at $g = 0$
- $G(t) = \mathcal{G}\left(\int N(g,t)dg\right)$

D. Cai, Shelley, McLaughlin, Rangan, L. Tao, Kovacic, Ly, Trnachina...

# Voltage-conductance I&F

$$\frac{\partial}{\partial t}p(v,g,t)+\frac{\partial}{\partial v}\left[\left(-g_L(V_L-v)+g(V_E-v)\right)p(v,g,t)\right]$$

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**Theorem (D. Salort, BP)**

- Stationary solutions belong to $L^{\frac{8}{7}^-}$
- Evolution solutions are globally bounded in $L^p$ (no blow-up)

**Open questions :** Long time asymptotic, regularity

# CONCLUSION

- Single neurone models are numerous and complex
- They share the property to describe excitability
- The I&F model is derived has a double Slow-Fast limit
- PDEs come as Eulerian versions