Time dependent mean-field games

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Mean-field games (MFG) systems

1. Differential games with a (very) large number of rational, indistinguishable and intelligent players;
2. Introduced in the trailblazing work of J-M. Lasry and P-L. Lions and M. Huang, P. Caines and R. Malhamé;
3. (Stochastic) Optimal control problem coupled with the transport of a density through the mean-field hypothesis and the feedback optimal control (Nash equilibrium).

A model MFG system is the following:

$$\begin{cases} -u_t + H(x, Du) = \Delta u + g[m] & \text{on } \mathbb{T}^d \times [0,T] \\ m_t - \operatorname{div}(D_p H m) = \Delta m & \text{on } \mathbb{T}^d \times [0,T], \end{cases} \tag{1}$$

equipped with the initial-terminal boundary conditions

$$\begin{cases} u(x,T) = u_T(x) \\ m(x,0) = m_0(x), \end{cases} \tag{2}$$

where

1. $g[m]$ is the mean-field coupling,
2. $\mathbb{T}^d$ is the d-dimensional torus, and
3. $T > 0$ is a fixed terminal instant.

Motivation for the MFG problem

The (stochastic) optimal control setting

Consider

$$\begin{cases} d\mathbf{x}_t = \mathbf{v}dt + \sqrt{2}dW_t \\ \mathbf{x}_0 = \mathbf{x}, \end{cases} \tag{3}$$

and

$$J(\mathbf{x}, \mathbf{v}, m) = \mathbb{E}^{\mathbf{x}} \left[\int_0^T L(\mathbf{x}_s, \mathbf{v}_s) + g[m](\mathbf{x}, s)ds + u_T(\mathbf{x}_T) \right]. \tag{4}$$

We know that a (viscosity) solution to (1) is the value function of the optimal control problem described by (3)-(4). Also, the feedback optimal control is given by

$$\mathbf{v}^* = -D_p H(\mathbf{x}, Du).$$

The mass-transport problem

The population governed by (3) evolves according to

$$m_t + \operatorname{div}(\mathbf{v}m) = \Delta m.$$

However, every agent knows that the feedback optimal control is given by

$$\mathbf{v}^* = -D_p H(\mathbf{x}, Du).$$

The former equation becomes then

$$m_t - \operatorname{div}(D_p H m) = \Delta m.$$

Main Assumptions

- We assume that g is a local power-like non-linearity, i.e.,
$$g[m](x,t) \doteq m^\alpha(x,t); \tag{5}$$
- The Hamiltonian H is supposed to be subquadratic, i.e.,
$$H(x,p) \leq C|p|^\gamma + C,$$
$$|D_p H| \leq C|p|^{\gamma-1} + C,$$
for $1 < \gamma < 2$;
- The Hamiltonian H is supposed to be superquadratic, i.e.,
$$C_1|p|^{2+\mu} + C_1 \leq H(x,p) \leq C_2|p|^{2+\mu} + C_2,$$
$$|D_p H|^2 \leq C|p|^\mu H + C,$$
for $0 < \mu < 1$;
- u_T and m_0 are supposed to be smooth functions on $\mathbb{T}^d$ and $m_0 \geq \kappa_0$, where $\kappa_0 > 0$;
- We also consider additional technical assumptions, which include a wide class of problems and examples.

Model Hamiltonians

The following are typical model Hamiltonians satisfying our Assumptions:

1. Subquadratic case:
$$H_s(x,p) = a(x)\left(1 + |p|^2\right)^{\frac{\gamma}{2}} + V(x).$$
2. Superquadratic case:
$$H_S(x,p) = a(x)\left(1 + |p|^2\right)^{\frac{2+\mu}{2}} + V(x).$$

Previous results

Existence of solutions - stationary case

1. Existence of weak solutions,
   - J-M. Lasry and P-L. Lions, 2006.
2. Existence of smooth solutions,
   - D. Gomes, G. Pires and H. Sánchez-Morgado, 2012;
   - D. Gomes and H. Sánchez-Morgado, 2013;
   - D. Gomes, S. Patrizi and V. Voskanyan, 2013.

Existence of solutions - time dependent case

1. Existence of weak solutions to (1)-(2),
   - J-M. Lasry and P-L. Lions, 2006.
2. Existence of weak solutions to the planning problem,
   - A. Porreta, 2013.
3. Smooth solutions for quadratic Hamiltonians,
   - P. Cardaliaguet, J-M. Lasry, P-L. Lions and A. Porreta, 2012.
4. Hamiltonians with quadratic or subquadratic growth and $g[m] = m^\alpha$:
   - Existence of smooth solutions for $\alpha > 0$ provided that
   $$\gamma \in \left(1, 1 + \frac{1}{d+1}\right).$$
   - Existence of smooth solutions for
   $$\alpha < \frac{2}{d-2},$$
   provided that
   $$\gamma \in \left(1 + \frac{1}{d+1}, 2\right).$$
   - P-L. Lions, 2012

Main Results

Subquadratic case

Theorem 1 (Gomes-P.-Sánchez-Morgado, *Comm. in PDE*, 2014)**.** *Let g be given as in* (5) *and assume that H is subquadratic. Then, there exists a classical solution (u,m) for* (1) *with the initial-terminal boundary conditions* (2)*, provided that*

$$\alpha < \alpha_{\gamma,d},$$

where

$$\alpha_{\gamma,d} > \frac{2}{d-2}.$$

In particular, we have:

$$\lim_{\gamma \to 1} \alpha_{\gamma,d} = +\infty,$$

and

$$\lim_{\gamma \to 2} \alpha_{\gamma,d} = \frac{2}{d-2}.$$

Superquadratic case

Theorem 2 (Gomes-P.-Sánchez-Morgado, 2013)**.** *Let g be given as in* (5) *and assume that H is superquadratic. Then, there exists a classical solution (u,m) for* (1) *with the initial-terminal boundary conditions* (2)*, provided that*

$$\alpha < \frac{2}{d(1+\mu) - 2}.$$

Notice that

$$\lim_{\mu \to 0} \alpha_{\mu,d} = \frac{2}{d-2},$$

and

$$\lim_{\mu \to 1} \alpha_{\mu,d} = \frac{1}{d-1}.$$

Strategy of the proofs

A regularization argument

To prove these results, a regularization of (1) is considered. It is done by replacing $g[m]$ by the nonlocal operator

$$g_\epsilon[m] = \eta_\epsilon * g[\eta_\epsilon * m],$$

where η_ϵ is a standard mollifying kernel, which in particular is symmetric. This yields the system

$$\begin{cases} -u_t^\epsilon + H(x, Du^\epsilon) = \Delta u^\epsilon + g_\epsilon[m^\epsilon] \\ m_t^\epsilon - \operatorname{div}(D_p H m^\epsilon) = \Delta m^\epsilon. \end{cases} \tag{6}$$

We use the convention $g_0 = g$

Subquadratic case

Theorem 3 (Polynomial estimates for the Fokker-Planck equation)**.** *Let (u^ϵ, m^ϵ) be a solution of* (6) *and $\|m^\epsilon\|_{L^\infty([0,T],L^{\beta_0}(\mathbb{T}^d))} \leq C$, for some $\beta_0 \geq 1$. Suppose further that $p > \frac{d}{2}$ and*

$$r = \frac{p(d(\theta-1)+2)}{2p-d}.$$

Then,

$$\int_{\mathbb{T}^d} (m^\epsilon)^{\beta_n}(\tau, x)\, dx \leq C + C \left\| |D_p H|^2 \right\|_{L^r\left(0,T;L^p\left(\mathbb{T}^d\right)\right)}^{r_n},$$

where

$$r_n = r\frac{\theta^n - 1}{\theta - 1},\ \theta > 1,$$

and

$$\beta_n = \theta^n \beta_0.$$

Lemma 1 (Upper bounds for the Hamilton-Jacobi equation)**.** *Let (u^ϵ, m^ϵ) be a solution of* (6) *and assume that H is subquadratic. Let $a,\ b > 1$ satisfy*

$$\frac{d}{2} < \frac{b(a-1)}{a}.$$

Then there exists $C > 0$ such that

$$\|u^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \leq C + C\|g_\epsilon(m)\|_{L^a(0,T;L^b(\mathbb{T}^d))}.$$

Theorem 4 (Gagliardo-Nirenberg inequality)**.** *Let (u^ϵ, m^ϵ) be a solution of* (6) *and assume that H is subquadratic. For $1 < p, r < \infty$ there are positive constants c and C such that*

$$\|D^2 u^\epsilon\|_{L^r(0,T;L^p(\mathbb{T}^d))} \leq c\|g_\epsilon(m^\epsilon)\|_{L^r(0,T;L^p(\mathbb{T}^d))} + c\|u^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{\frac{\gamma}{2-\gamma}} + C$$

.

Superquadratic case

Theorem 5 (Polynomial estimates for the Fokker-Planck equation)**.** *Let (u^ϵ, m^ϵ) be a solution of* (6)*. Assume that H is superquadratic. Assume further that $0 < \mu < 1 < \beta_0,\ \theta,\ p,\ r,$ and $0 \leq \upsilon \leq 1$ satisfy*

$$\alpha p = \frac{\theta^n \beta_0}{\theta^n + \upsilon - \theta^n \upsilon},$$

and

$$r = \frac{d(\theta-1)+2}{2}.$$

Then

$$\|g_\epsilon\|_{L^\infty(0,T;L^p(\mathbb{T}^d))} \leq C + C\left\|Du^\epsilon\right\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{\frac{(2+2\mu)(\theta^n-1)r\upsilon\alpha}{\theta^n\beta_0(\theta-1)}}.$$

Lemma 2 (Upper bounds for the Hamilton-Jacobi equation)**.** *Suppose (u^ϵ, m^ϵ) is a solution of* (6) *and H is superquadratic. Then, if*

$$p > \frac{d}{2},$$

$$\|u^\epsilon\|_{L^\infty(\mathbb{T}^d \times [0,T])} \leq C + C\|g_\epsilon(m)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}.$$

Furthermore, if

$$\frac{1}{r} + \frac{1}{s} = \frac{1}{p} + \frac{1}{q} = 1$$

and

$$\frac{p}{s} > \frac{d}{2},$$

we have

$$\|u^\epsilon\|_{L^\infty(\mathbb{T}^d \times [0,T])} \leq C + C\|g_\epsilon(m)\|_{L^r(0,T;L^p(\mathbb{T}^d))}.$$

Theorem 6 (Estimates by the non-linear adjoint method)**.** *Suppose that H is superquadratic. Let (u^ϵ, m^ϵ) be a solution of* (6) *and assume that $p > d$. Then*

$$\|Du^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \leq C + C\|g_\epsilon(m)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{\frac{1}{1-\mu}} + C\|g_\epsilon(m)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{\frac{1}{1-\mu}}\|u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{\frac{1}{1-\mu}}.$$

Further regularity

Bootstrapping regularity and passing to the limit

1. Lipschitz regularity for u^ϵ
2. Bounds for m^ϵ from below
3. Regularity for (u^ϵ, m^ϵ) in Sobolev spaces
4. Additional estimates allowing one to pass to the limit $\epsilon \to 0$

Regularity for the Fokker-Planck equation in Sobolev spaces

Corollary 1. *Let (u^ϵ, m^ϵ) be a solution of* (6) *with initial-terminal conditions* (2)*. Assume that $g[m] = m^\alpha$ and that H is either sub or superquadratic. Then*

- $D^2_{xx}m^\epsilon, m^\epsilon_t \in L^2(\mathbb{T}^d \times [0,T])$*, and* $D_x m^\epsilon \in L^\infty([0,T], L^2(\mathbb{T}^d))$*;*
- $D^3_{xxx}m^\epsilon, D^2_{xt}m^\epsilon \in L^2(\mathbb{T}^d \times [0,T])$ $D^2_{xx}m^\epsilon \in L^\infty([0,T], L^2(\mathbb{T}^d))$ *and*
- *there is $r > d$ such that $D_x m^\epsilon, D^2_{xx}m^\epsilon, m^\epsilon_t \in L^r(\mathbb{T}^d \times [0,T])$ and then $m^\epsilon \in C^{0,1-d/r}(\mathbb{T}^d \times [0,T])$.*

Regularity by the Hopf-Cole transformation

Consider the following Hopf-Cole transformation:

$$w \doteq \ln m^\epsilon.$$

Lemma 3. *Let (u^ϵ, m^ϵ) be a solution of* (6) *with initial-terminal conditions* (2)*. Assume that $g[m] = m^\alpha$ and that H is either sub or superquadratic. Then $\ln m^\epsilon$ is Lipschitz and, therefore, m^ϵ is bounded by above and below.*

In particular the Lemma ensures that m^ϵ is bounded away from zero

Regularity for the Hamilton-Jacobi equation

Lemma 4. *Let (u^ϵ, m^ϵ) be a solution of* (6) *with initial-terminal conditions* (2)*. Assume that $g[m] = m^\alpha$ and that H is either sub or superquadratic. Then*

- $D^2_{xx}u^\epsilon, u^\epsilon_t \in L^r(\mathbb{T}^d \times [0,T])$*, for any $r < \infty$;*
- $D^3_{xxx}u^\epsilon, D^2_{xt}u^\epsilon \in L^2(\mathbb{T}^d \times [0,T])$ $D_{xx}u^\epsilon \in L^\infty([0,T], L^2(\mathbb{T}^d))$*;*
- *There exists $\gamma \in (0,1)$ such that $u^\epsilon \in \mathcal{C}^{0,\gamma}\left(\mathbb{T}^d \times [0,T]\right)$.*

Passing to the limit

We observe that

1. There exists u such that $u^\epsilon \to u$ in $\mathcal{C}^{0,\gamma}\left(\mathbb{T}^d \times [0,T]\right)$ through some subsequence, uniformly in compacts;
2. Compactness imply that we also have $Du^\epsilon \to Du$;
3. There exists m such that $m^\epsilon \to m$ in $\mathcal{C}^{0,\gamma}\left(\mathbb{T}^d \times [0,T]\right)$ through some subsequence, uniformly in compacts.

Follow-up work: forward-forward mean-field games

Forward-forward mean-field games

- Long-time behavior of MFG systems, $T \to \infty$
- Convergence to the equilibrium problem - numerical methods
  - Reverse time in the Hamilton-Jacobi equation
  - Prescribe initial-initial conditions for the MFG systems

Consider the following variation of the original MFG system

$$\begin{cases} u_t + H(x, Du) = \Delta u + g[m] & \text{on } \mathbb{T}^d \times [0,T] \\ m_t - \operatorname{div}(D_p H m) = \Delta m & \text{on } \mathbb{T}^d \times [0,T], \end{cases} \tag{7}$$

equipped with *initial-initial* boundary conditions:

$$\begin{cases} u(x,0) = u_0(x) \\ m(x,0) = m_0(x). \end{cases} \tag{8}$$

- Main difficulty: Fokker-Planck is no longer the (formal) adjoint equation to the Hamilton-Jacobi

A result on existence of classical solutions

Theorem 7 (Gomes-P.)**.** *Let g be given as in* (5) *and assume that H is subquadratic. Then, there exists a classical solution (u,m) for* (7) *with the initial-initial boundary conditions* (8)*, provided that*

$$\alpha < \alpha_{\gamma,d},$$

where

$$\alpha_{\gamma,d} > \frac{2}{d}.$$

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