From the Boltzmann equation to incompressible viscous hydrodynamics

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Fluid dynamics

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Particle number density: $F(t,x,v) \geq 0$

$(t,x,v) \in [0,\infty) \times \Omega \times \mathbb{R}^D$
$\Omega \subset \mathbb{R}^D, \ D \geq 2 \ (D=1)$

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The incompressible Navier-Stokes-Fourier system:

$$\begin{aligned} \partial_t u + u \cdot \nabla_x u - \nu \Delta_x u &= -\nabla_x p \\ \nabla_x \cdot u &= 0 \\ \frac{D+2}{2} \left(\partial_t \theta + u \cdot \nabla_x \theta \right) - \kappa \Delta_x \theta &= 0 \end{aligned}$$

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The Boltzmann equation:

$$\left(\partial_t + v \cdot \nabla_x \right) F(t,x,v) = \mathcal{B}(F,F)(t,x,v)$$

The Boltzmann collision operator

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$$\boxed{\mathcal{B}\left(F,F\right)\left(t,x,v\right)=\int_{\mathbb{R}^D}\int_{\mathbb{S}^{D-1}}\left(F'F'_*-FF_*\right)b\left(v-v_*,\sigma\right)d\sigma dv_*}$$

$$F'=F(t,x,v'),\quad F'_*=F(t,x,v'_*),\quad F_*=F(t,x,v_*)$$

$$v'=\tfrac{v+v_*}{2}+\tfrac{|v-v_*|}{2}\sigma,\qquad v'_*=\tfrac{v+v_*}{2}-\tfrac{|v-v_*|}{2}\sigma$$

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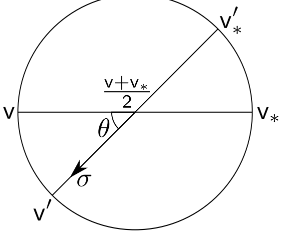
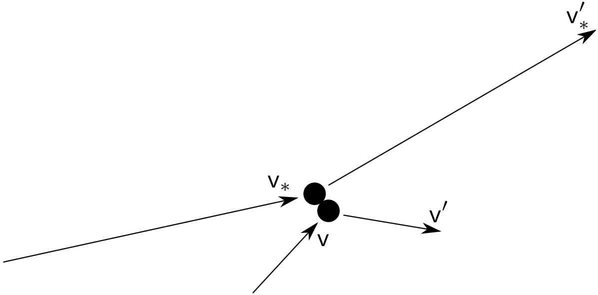
$$\begin{cases} v+v_*=v'+v'_* & \text{(conservation of momentum)}\\ |v|^2+|v_*|^2=|v'|^2+|v'_*|^2 & \text{(conservation of energy)}\end{cases}$$





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5 hypotheses:

- binary collisions (rarefied gas)
- localization in time and space of collisions
- elastic collisions
- micro-reversibility of collisions
- molecular chaos

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<u>The collision kernel:</u> $\quad b(v-v_*,\sigma) = b(\left|v-v_*\right|, \cos\theta) \geq 0$

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- Intermolecular forces deriving from an inverse power potential:

$$\phi(r)=\frac{1}{r^{s-1}},\qquad s>2$$

$$b(v-v_*,\sigma)=|v-v_*|^\gamma\, b_0(\cos\theta),\qquad \gamma=\frac{s-5}{s-1}$$

$$\sin^{D-2}\theta\, b_0(\cos\theta)\approx\frac{1}{\theta^{1+\nu}}\notin L^1_{\mathrm{loc}},\qquad \nu=\frac{2}{s-1}$$

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long-range interactions $\Rightarrow$ grazing collisions $\Rightarrow$ non-integrable kernel

$$\boxed{\left(\partial_t + v \cdot \nabla_x\right) F(t,x,v) = \mathcal{B}\left(F,F\right)(t,x,v)}$$

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Macroscopic variables:

- density: $\rho(t,x) = \int_{\mathbb{R}^D} F(t,x,v)\; dv$
- bulk velocity: $\rho u(t,x) = \int_{\mathbb{R}^D} F(t,x,v) v\; dv$
- temperature: $\rho\theta(t,x) = \int_{\mathbb{R}^D} F(t,x,v) \frac{|v-u(t,x)|^2}{D}\; dv$

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Microscopic conservation laws: $\int_{\mathbb{R}^D} \mathcal{B}(F,F)(t,x,v) \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{pmatrix} dv = 0$

$$\boxed{(\partial_t + v \cdot \nabla_x)\, F(t,x,v) = \mathcal{B}\,(F,F)\,(t,x,v)}$$

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Macroscopic conservation laws:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + P) = 0 \\ \partial_t \left(\rho \frac{|u|^2}{2} + \frac{D}{2}\rho\theta\right) + \nabla_x \cdot \left(\left(\rho\frac{|u|^2}{2} + \frac{D}{2}\rho\theta\right) u + Pu + q\right) = 0 \end{cases}$$

- stress tensor: $P(t,x) = \int_{\mathbb{R}^D} F(t,x,v)(v-u) \otimes (v-u)\, dv$
- thermal flux: $q(t,x) = \int_{\mathbb{R}^D} F(t,x,v)(v-u)|v-u|^2\, dv$

$$\boxed{(\partial_t + v \cdot \nabla_x) F_\epsilon(t,x,v) = \underset{\substack{\uparrow \\ \text{Knudsen number} \\ \approx\text{mean free path}}}{\frac{1}{\epsilon}} \mathcal{B}(F_\epsilon, F_\epsilon)(t,x,v)}$$

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<u>Hyperbolic scaling:</u> $F_\epsilon(t,x,v) = F\left(\frac{t}{\epsilon},\frac{x}{\epsilon},v\right)$

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Continuum limit $\epsilon \to 0$: $\quad F_\epsilon \to F \quad \Rightarrow \quad \mathcal{B}(F,F) = 0$

$\Rightarrow \quad F = \dfrac{\rho}{(2\pi\theta)^{\frac{D}{2}}} e^{-\frac{|v-u|^2}{2\theta}}$ is a Maxwellian $\quad \Rightarrow \quad P = \mathrm{Id}\rho\theta, \quad q = 0$

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Compressible Euler system:

$$\begin{cases} \partial_t\rho + \nabla_x\cdot(\rho u) = 0 \\ \partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u) + \nabla_x(\rho\theta) = 0 \\ \partial_t\left(\rho\frac{|u|^2}{2} + \frac{D}{2}\rho\theta\right) + \nabla_x\cdot\left(\left(\rho\frac{|u|^2}{2} + \frac{D+2}{2}\rho\theta\right)u\right) = 0 \end{cases}$$

$$\boxed{\Big(\underset{\substack{\uparrow\\ \text{Strouhal}\\ \text{number}}}{\epsilon}\,\partial_t + v\cdot\nabla_x\Big)F_\epsilon(t,x,v) = \frac{1}{\underset{\substack{\uparrow\\ \text{Knudsen number}\\ \approx\text{mean free path}}}{\epsilon}}\mathcal{B}\left(F_\epsilon,F_\epsilon\right)(t,x,v)}$$

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Parabolic scaling: $F_\epsilon(t,x,v) = F\left(\frac{t}{\epsilon^2},\frac{x}{\epsilon},v\right)$

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Parabolic scaling: $F_\epsilon(t,x,v) = F\left(\frac{t}{\epsilon^2},\frac{x}{\epsilon},v\right)$

Fluctuations: $F_\epsilon = M(v)(1+\underset{\substack{\uparrow\\ \text{Mach}\\ \text{number}}}{\epsilon}\,g_\epsilon)$ where $M(v) = \frac{1}{(2\pi)^{\frac{D}{2}}}e^{-\frac{|v|^2}{2}}$

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Continuum limit $\epsilon \to 0$: $\quad g_\epsilon \to g$

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<u>Continuum limit $\epsilon \to 0$:</u> $g_\epsilon \to g$

<u>Order ϵ^{-1}:</u> $\mathcal{L}(g) = 0 \Rightarrow g = \rho + v \cdot u + \left(\frac{|v|^2}{2} - \frac{D}{2}\right)\theta$ is an infinitesimal Maxwellian

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Order 1: $\boxed{\nabla_x \cdot u = 0}$ and $\nabla_x(\rho+\theta) = 0$

$\Rightarrow \boxed{\rho + \theta = 0}$ and $\boxed{g = v\cdot u + \left(\frac{|v|^2}{2} - \frac{D+2}{2}\right)\theta}$

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Order ϵ:

$$\begin{cases} \partial_t \int_{\mathbb{R}^D} g_\epsilon v M dv + \frac{1}{\epsilon}\nabla_x \cdot \int_{\mathbb{R}^D} g_\epsilon A M dv = -\frac{1}{\epsilon}\nabla_x \int_{\mathbb{R}^D} g_\epsilon \frac{|v|^2}{D} M dv \\ \partial_t \int_{\mathbb{R}^D} g_\epsilon \left(\frac{|v|^2}{2} - \frac{D+2}{2}\right) M dv + \frac{1}{\epsilon}\nabla_x \cdot \int_{\mathbb{R}^D} g_\epsilon B M dv = 0 \end{cases}$$

where $\quad A = v\otimes v - \frac{|v|^2}{D}\mathrm{Id} \quad$ and $\quad B = \left(\frac{|v|^2}{2} - \frac{D+2}{2}\right)v.$

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Idea: use that $\mathcal{L}$ is Fredholm (index 0) and self-adjoint on $L^2\left(Mdv\right)$, so that $A, B \in \mathrm{Im}\left(\mathcal{L}\right) \Rightarrow A = \mathcal{L}\left(\bar{A}\right)$, $B = \mathcal{L}\left(\bar{B}\right)$.

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Handling the fluxes:

$$\begin{aligned}
&\frac{1}{\epsilon}\nabla_x\cdot\int_{\mathbb{R}^D} g_\epsilon A M dv \\
&= \frac{1}{\epsilon}\nabla_x\cdot\int_{\mathbb{R}^D} g_\epsilon \mathcal{L}\left(\bar{A}\right) M dv \\
&= \nabla_x\cdot\int_{\mathbb{R}^D} \frac{1}{\epsilon}\mathcal{L}\left(g_\epsilon\right)\bar{A} M dv \\
&= \nabla_x\cdot\int_{\mathbb{R}^D} \mathcal{Q}\left(g_\epsilon, g_\epsilon\right)\bar{A} M dv - \nabla_x\cdot\int_{\mathbb{R}^D}(\epsilon\partial_t + v\cdot\nabla_x)g_\epsilon\bar{A} M dv
\end{aligned}$$

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We finally obtain in the limit:

$$\begin{cases} \partial_t u + u\cdot\nabla_x u - \nu\Delta_x u = -\nabla_x p \\ \frac{D+2}{2}\left(\partial_t\theta + u\cdot\nabla_x\theta\right) - \kappa\Delta_x\theta = 0 \\ \nabla_x\cdot u = 0 \end{cases}$$

with

$$\nu = \frac{1}{(D-1)(D+2)}\int_{\mathbb{R}^D} A : \mathcal{L}(A)Mdv$$
$$\kappa = \frac{1}{D}\int_{\mathbb{R}^D} B\cdot\mathcal{L}(B)Mdv$$

$$\boxed{\begin{gathered}(\epsilon^s \partial_t + v \cdot \nabla_x) F_\epsilon(t,x,v) = \frac{1}{\epsilon^q} \mathcal{B}\left(F_\epsilon, F_\epsilon\right)(t,x,v) \\ F_\epsilon = M(v)(1+\epsilon^m g_\epsilon)\end{gathered}}$$

1. Compressible Euler ($q = 1,\ s = 0,\ m = 0$)
2. Acoustic waves ($q = 1,\ s = 0,\ m > 0$)
$$\begin{cases} \partial_t \rho + \nabla_x \cdot u = 0, \qquad \partial_t(\rho + \theta) + \frac{D+2}{2} \nabla_x \cdot u = 0 \\ \partial_t u + \nabla_x(\rho + \theta) = 0 \end{cases}$$
3. Incompressible Navier-Stokes-Fourier ($q = 1,\ s = 1,\ m = 1$)
4. Incompressible Stokes-Fourier ($q = 1,\ s = 1,\ m > 1$)
$$\begin{cases} \partial_t u - \nu \Delta_x u = -\nabla_x p, \qquad \nabla_x \cdot u = 0 \\ \partial_t \theta - \kappa \Delta_x \theta = 0 \end{cases}$$
5. Incompressible Euler-Fourier ($q > 1,\ s = 1,\ m = 1$)
$$\begin{cases} \partial_t u + u \cdot \nabla_x u = -\nabla_x p, \qquad \nabla_x \cdot u = 0 \\ \partial_t \theta + u \cdot \nabla_x \theta = 0 \end{cases}$$

$$\boxed{\begin{gathered}(\epsilon^s \partial_t + v \cdot \nabla_x) F_\epsilon(t,x,v) = \frac{1}{\epsilon^q} \mathcal{B}\left(F_\epsilon, F_\epsilon\right)(t,x,v) \\ F_\epsilon = M(v)(1 + \epsilon^m g_\epsilon)\end{gathered}}$$

$$\boxed{\begin{gathered}(\epsilon^s \partial_t + v \cdot \nabla_x) F_\epsilon(t,x,v) = \frac{1}{\epsilon^q} \mathcal{B}\left(F_\epsilon, F_\epsilon\right)(t,x,v) \\ F_\epsilon = M(v)(1 + \epsilon^m g_\epsilon)\end{gathered}}$$

It is not possible to obtain
a compressible Navier-Stokes-Fourier system!

<u>von Kármán relation:</u>

$$\text{Reynolds} \approx \frac{\text{Mach}}{\text{Knudsen}}$$

From Boltzmann to Navier-Stokes

Theorem (Ars., Bardos, Golse, Levermore, Lions, Masmoudi, Saint-Raymond, '91 - '11)

We consider:

- *a collision kernel which derives from an inverse power potential.*
- *renormalized solutions* $F_\epsilon = M(1+\epsilon g_\epsilon) \in L^\infty_t L^1_{x,v}$ *of*

$$(\epsilon \partial_t + v \cdot \nabla_x) g_\epsilon = -\frac{1}{\epsilon} \mathcal{L}(g_\epsilon) + \mathcal{Q}(g_\epsilon, g_\epsilon)$$

  with well-prepared initial data.

Then, as $\epsilon \to 0$*:*

- g_ϵ *is weakly relatively compact in* $L^1\left((1+|v|^2)Mdtdxdv\right)$.
- *each limit point* g *of* g_ϵ *satisfies* $g = v \cdot u + \left(\frac{|v|^2}{2} - \frac{D+2}{2}\right)\theta$ *where* $(u,\theta) \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ *is a Leray solution of the incompressible Navier-Stokes-Fourier system.*

Mathematical difficulties

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$$\boxed{(\epsilon\partial_t + v\cdot\nabla_x)g_\epsilon = -\frac{1}{\epsilon}\mathcal{L}(g_\epsilon) + \mathcal{Q}(g_\epsilon, g_\epsilon)}$$

1. Lack of a priori estimates! We only have the entropy inequality.
2. Renormalized solutions: $g_\epsilon \in L^1_{\text{loc}}$ (at most $L\log L$) such that

$$(\epsilon\partial_t + v\cdot\nabla_x)\frac{g_\epsilon}{2+\epsilon g_\epsilon} = -\frac{2}{\epsilon(2+\epsilon g_\epsilon)^2}\mathcal{L}(g_\epsilon) + \frac{2}{(2+\epsilon g_\epsilon)^2}\mathcal{Q}(g_\epsilon, g_\epsilon)$$

3. The macroscopic conservation laws are not known to hold for renormalized solutions.
4. Time compactness: there are oscillations!
5. Space compactness: there is some compactness thanks to velocity averaging lemmas. Moreover, we need the nonlinear weak compactness estimate:

$$\frac{g_\epsilon^2}{2+\epsilon g_\epsilon} \quad \text{is weakly compact in } L^1_{\text{loc}}.$$