Orthogonal polynomials and diffusions

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Motivations

- Describe natural bases in $\mathcal{L}^2(\mu)$ where computations are easy to made.
- Describe some measures μ hard to handle in high dimensions through formal manipulations : in particular compute moments.
- Describe examples of Markov diffusions where one may compute explicitly the spectral decomposition, and hence heat kernel measures, etc.
- Try to understand the underlying structure of sets on which such measure exist.
- Understand some specific properties of families of orthogonal polynomials : generating functions, associated Markov sequence problems, hypergroup properties, etc.

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Context

μ probability measure on $\mathbb{R}$ or $\mathbb{R}^d$ such that polynomials are dense in $\mathcal{L}^2(\mu)$.

Natural basis for $\mathcal{L}^2(\mu)$ given by orthogonal polynomials, obtained by orthonormalization of the sequence of monomials.

In dimension 1, orthonormalize the sequence $1, x, \ldots, x^n, \ldots$ to get a (unique up to the sign) sequence of polynomials P_n which are orthogonal and norm 1.

Not unique in higher dimension : for any k, a choice is made of a basis of the orthogonal complement of $\mathcal{P}_{k-1}$ in $\mathcal{P}_k$, where $\mathcal{P}_k$ is the space of polynomials with total degree $\leq k$.

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Most famous examples

On $\mathbb{R}$: Hermite polynomials : $\mu(dx) = e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$.

On $[0,\infty)$: Laguerre polynomials : $\mu(dx) = C_a x^a e^{-x} dx$.

On $[-1,1]$: Jacobi polynomials $\mu(dx) = C_{a,b}(1-x)^a(1+x)^b dx$.

In those three examples, the associated polynomials are also eigenvectors of Diffusion Operators, that is second order elliptic differential operators.

- Hermite case : $L(f) = f'' - xf'$, $LP_n = -nP_n$.
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Symmetric diffusion generators

Symmetry : $\int gL(f)d\mu = \int fL(g)d\mu$.

Diffusion : $L(\Phi(f_1, \cdots, f_k)) = \sum_i L(f_i)\partial_i\Phi + \sum_{ij}\Gamma(f_i, f_j)\partial^2_{ij}\Phi$,

$\Gamma(f_i, f_j) = \frac{1}{2}\Big(L(f_i f_j) - f_i L(f_j) - f_j L(f_i)\Big)$.

In particular $L(1) = 0$ and $\int L(f)d\mu = 0$ (invariance).

In $\mathbb{R}^n$, $\mu(dx) = \rho(x)dx$ then

$$L(f) = \frac{1}{\rho}\sum_{ij}\partial_i\Big(g^{ij}\rho\partial_j f\Big).$$

If L self adjoint and has discrete spectrum : another natural basis for $\mathcal{L}^2(\mu)$ is given by the eigen vectors of L.
We are looking for the situation where those basis coincide.

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For diffusion with polynomial eigenvectors

$L(f) = \sum_{ij} g^{ij}(x)\partial^2_{ij} f + \sum_i b^i(x)\partial_i f$

$L(x_i) = b^i(x), \ g^{ij}(x) = \Gamma(x^i, x^j).$

$\mathcal{P}_n$:= polynomials with total degree less than n. If there is a basis of $\mathcal{P}_n$ formed with eigenvectors for L then

$$L : \mathcal{P}_n \mapsto \mathcal{P}_n.$$

$b^i(x)$ polynomial degree ≤ 1

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$\int PL(Q)d\mu = \int QL(P)d\mu$ for any pair of polynomials.

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How to use it ? Moments

Computation of $\int x^n \, d\mu$ for the Gaussian measure :

$L_x = \partial_x^2 - x\partial_x$, $\mu(dx) = e^{-x^2/2}dx$.

$L(x^n) = n(n-1)x^{n-1} - nx^n$.

$$\int L(x^n)d\mu = 0 \implies \int x^n d\mu = (n-1)\int x^{n-2}d\mu.$$

Recurrence formula for the moments.

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Recurrence formula for the moments.

How to use it ? Eigenvectors

Complex representation for Hermite Polynomials

On $\mathbb{R}^2$, $L = L_x + L_y$ symmetric wrt $d\mu(x)d\mu(y)$.

$L(x+iy) = -(x+iy)$, $\Gamma(x+iy, x+iy) = 0$.

$$\begin{aligned} L(x+iy)^n = \quad & n(x+iy)^n L(x+iy) \\ & + n(n-1)(x+iy)^{n-2}\Gamma(x+iy, x+iy) \\ & = -n(x+iy)^n \end{aligned}$$

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General Problem

Find

- all regular open sets $\Omega \subset R^n$, (piecewise smooth boundary)
- all probability measures μ on Ω (with dense polynomials),
- all symmetric diffusion operators L on Ω,

such that $\mathcal{L}^2(\mu)$ has a orthonormal basis formed of eigenvectors for L which are polynomials.

We shall restrict to the elliptic case : $(g^{ij})(x)$ everywhere positive definite on Ω. In this case, the inverse matrix $g_{ij}(x)$ defines a Riemanian metric on Ω.

Problem invariant under affine transformations on Ω.

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No other examples (up to affine transformations) (Mazet, '97)

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Laplace operator on spheres $S^{n-1} \subset \mathbb{R}^n$.

$L(x_i) = -(n-1)x_i$.

$\Gamma(x_i, x_j) = \delta_{ij} - x_i x_j$.

$X := 2(x_1^2 + \cdots x_p^2) - 1$, $p \leq n$.

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$\frac{1}{4}L(\Phi(X)) = \hat{L}(\Phi)(X)$

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$\hat{L}$: Jacobi operator with parameters $a = (n-p)/2 + 1$, $b = p/2 + 1$.

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Jacobi to Hermite scale Jacobi on $(-\sqrt{n}, \sqrt{n})$, $a = b = n$, $n \to \infty$

Jacobi to Laguerre move and scale Jacobi on $(0, \sqrt{n})$, limit $a = n \to \infty$, b fixed.

Hermite to Laguerre Hermite on R^d, applied on $f(X)$ with $X = x_1^2 + \cdots x_d^2$: Laguerre with parameter $a = d/2 + 1$.

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Hermite to Laguerre Hermite on R^d, applied on $f(X)$ with $X = x_1^2 + \cdots x_d^2$: Laguerre with parameter $a = d/2 + 1$.

Geometric interpretation for Hermite and Laguerre

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Higher dimensional models

Few examples

- Dirichlet measures on the simplex $x_i \geq 0$, $\sum_i x_i \leq 1$.
- On the unit ball $\sum_i x_i^2 \leq 1$. $\mu(dx) = (1 - \|x\|^2)^a dx$ (Bargmann measures).
- Law of the spectrum of random matrices : GOE, GUE, $SO(n)$, $SU(n)$, $Sp(n)$, and many other on matrices. The variables are then the elementary symmetric functions of the eigenvalues.
- Root systems, Affine Hecque algebras (McDonald polynomials).
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Lie group actions

G compact group of matrices acting on $\mathbb{R}^d$ on a space of matrices

Examples : $g \mapsto Mg$, $g \mapsto g^* Mg$, etc.

$A \in \mathcal{L}(G)$ $(e^{tA} \in G)$ $X_A(F)(M) = \lim_{t \to 0} \frac{F(Me^{tA}) - F(M)}{t}$.

Then $X_A(F) = \sum_{ijk} A_{ik} M_{kl} \partial_{M_{kj}} F$.
$L = \sum X_{A_i}^2$ maps polynomials in the entries into polynomials in the entries, and is symmetric with respect to the Haar measure.
Example Laplace operators (Casimir operators) on compact groups (in general non elliptic).
Still true for functions which are invariant under actions of subgroups : main source of natural elliptic examples for models.

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In $\mathbb{R}^2$

In $\mathbb{R}^2$, up to affine transformations, we are able to describe all the sets Ω, all the measures and all the associated operators

- 11 compact sets Ω
- 7 non compact ones

For any of these Ω, there exists a least one measure for which the model comes from Lie group action.

For many values of parameters appearing in the measure, existence of geometric interpretations.

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The 11 compact models in dimension 2 : triangle

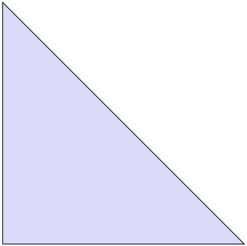

FIGURE: Triangle

Equation : $xy(1 - x - y) = 0$.
Measure $\rho(x) = x^a y^b (1 - x - y)^c$.

The 11 compact models in dimension 2 : circle

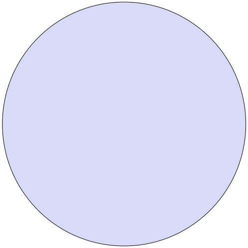

FIGURE: Circle

Equation : $(1 - x^2 - y^2) = 0$.
Measure $\rho(x) = (1 - x^2 - y^2)^a$.

The 11 compact models in dimension 2 : square

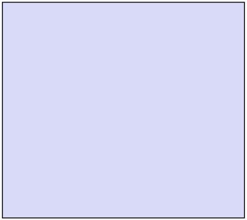

FIGURE: Square

Equation : $(1-x)(1+x)(1-y)(1+y) = 0.$
Measure $\rho(x) = (1-x)^a(1+x)^b(1-y)^c(1+y)^d.$

The 11 compact models in dimension 2 : double parabola

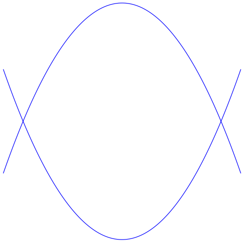

FIGURE: Coaxial Parabolas

Equation : $(y - x^2 + 1)(y - 1 + \alpha x^2) = 0$.
Measure $\rho(x) = (y - x^2 + 1)^a (y - 1 + \alpha x^2)^b$.

The 11 compact models in dimension 2 : Parabola with two lines 1

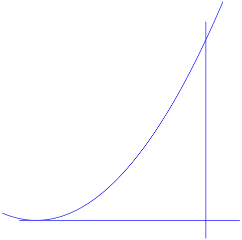

FIGURE: Parabola with two lines 1

Equation : $(y - x^2)y(1 - x) = 0$.
Measure $\rho(x) = (y - x^2)^a y^b (1 - x)^c$.

The 11 compact models in dimension 2 : Parabola with two lines 2

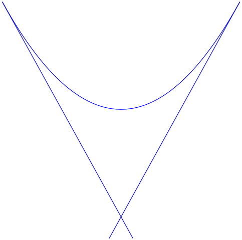

FIGURE: Parabola with two lines 2

Equation : $(y - x^2)(y + 2x + 1)(y - 2x + 1) = 0$.
Measure $\rho(x) = (y - x^2)^a(y + 2x + 1)^b(y - 2x + 1)^c$.

The 11 compact models in dimension 2 : Cuspidal Cubic 1

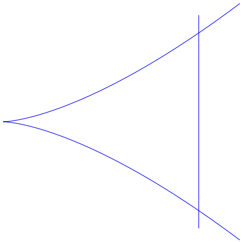

FIGURE: Cuspidal cubic 1

Equation : $(y^2 - x^3)(1 - x) = 0$.
Measure $\rho(x) = (y^2 - x^3)^a (1 - x)^b$.

The 11 compact models in dimension 2 : Cuspidal Cubic 2

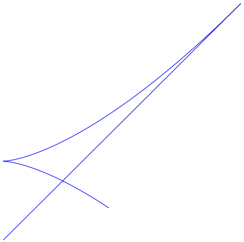

FIGURE: Cuspidal cubic 2

Equation : $(y^2 - x^3)(2y - 3x + 2) = 0.$
Measure $\rho(x) = (y^2 - x^3)^a(2y - 3x + 2)^b.$

The 11 compact models in dimension 2 : Nodal Cubic

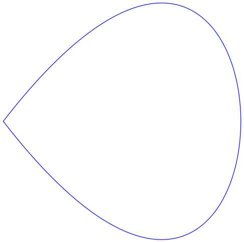

Figure: Nodal Cubic

Equation : $y^2 - x^2(1-x) = 0$.
Measure $\rho(x) = (y^2 - x^2(1-x))^a$.

The 11 compact models in dimension 2 : Swallow Tail

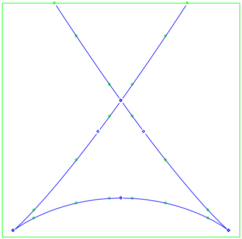

FIGURE: Swallow Tail

Equation :$4\,x^2 - 27\,x^4 + 16\,y - 128\,y^2 - 144\,x^2 y + 256\,y^3 = 0$

Measure

$\rho(x) = (4\,x^2 - 27\,x^4 + 16\,y - 128\,y^2 - 144\,x^2 y + 256\,y^3)^a.$

The 11 compact models in dimension 2 : Deltoid

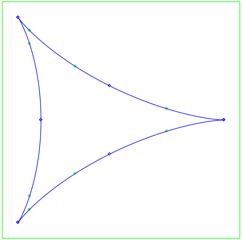

FIGURE: Deltoid

Equation : $(x^2+y^2)^2+18(x^2+y^2)-8x^3+24xy^2-27=0.$
Measure
$\rho(x)=\Big((x^2+y^2)^2+18(x^2+y^2)-8x^3+24xy^2-27\Big)^a.$

Comments

- Boundaries of Ω are algebraic curves with degree less than 4
- The measures are related to the equations of the boundaries : every irreducible factor to some power
- When the boundary is degree 4, the associated metric has constant curvature
- Curvature is 0 for square, parabola with two tangents, and deltoid.
- Curvature is constant positive in every other case.
- In circle and triangle case, the metric $g^{ij}(x)$ is not unique.
- Unique in every other case
- For nodal cubic, the metric is unique, the curvature is not constant, but when $a = -1/2$, it has a natural interpretation coming from the 4-d sphere (twisted Hopf fibration).

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More comments

- Most of the 2-d equations describing boundaries are discriminants (the set where some polynomial of a certain type has two coinciding roots).
- Every model has a geometric representation when the exponents in the measure are set to $-1/2$.
- Many other geometric interpretation for exponents half integers.
- From the Riemanian geometric point of view, those half integers measure do not always correspond to wrapped products.
- There is a relationship between the type of the singular points of the model and the angles of the boundaries of the cells it comes from in the geometric interpretation : ordinary double points correspond to $\pi/2$, cusps to $\pi/3$, and double tangents to $\pi/4$.
- Every two dimensional model has a natural

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- Many other geometric interpretation for exponents half integers.
- From the Riemanian geometric point of view, those half integers measure do not always correspond to wrapped products.
- There is a relationship between the type of the singular points of the model and the angles of the boundaries of the cells it comes from in the geometric interpretation : ordinary double points correspond to $\pi/2$, cusps to $\pi/3$, and double tangents to $\pi/4$.
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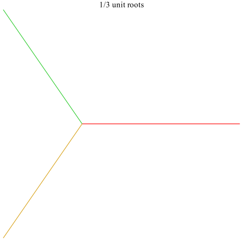

FIGURE: $1, j, \bar{j}$

$Z : \mathbb{C} = \mathbb{R}^2 \mapsto \mathbb{C} = \mathbb{R}^2$

$$Z = e^{i1.z} + e^{ij.z} + e^{i\bar{j}.z}.$$

The image of $\mathbb{R}^2$ under Z is the Deltoid domain.

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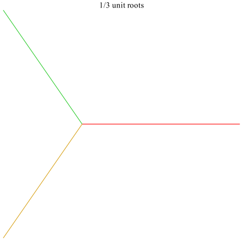

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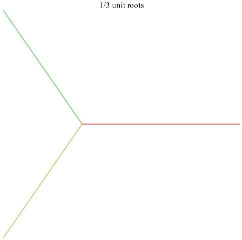

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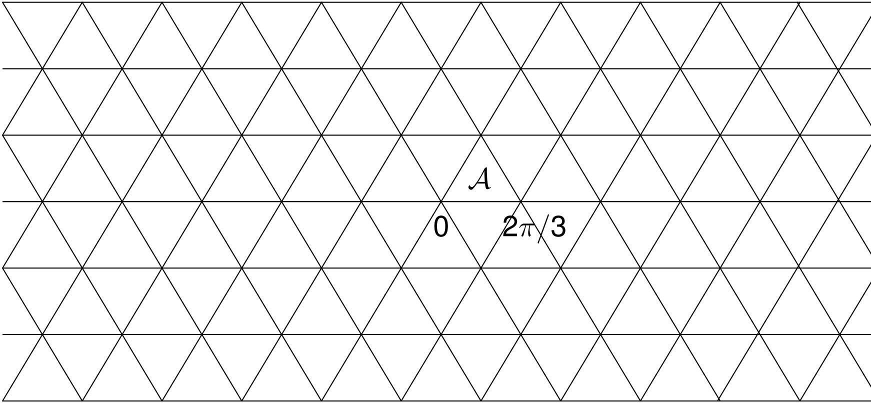
Triangular lattice



Z is invariant under the symmetries of the triangular lattice.
Z is injective on each cell of the lattice.

Deltoid Continued

Any function which has the symmetries of the triangular lattice is a function of Z.

With Δ usual Laplacian in $\mathbb{R}^2$

$$\Delta\Big(f(Z)\Big) = L_{-1/2}(f)(Z).$$

$L_{-1/2}$: 2-d Laplace operator on functions having the symmetries of the triangular lattice. ρ is the image measure of the Lebesgue measure through Z.

Other interpretation for $p = 1$ Laplace of $SU(3)$ on spectral functions. Z is then the trace of the matrix $M \in SU(3)$.

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Non compact cases

Any kind of products of intervals (5 different models)

In addition : above a parabola or to the right of the cuspidal cubic

Measures : same as before with some exponential factors (similar to the Laguerre case)

When no boundaries : only the Gaussian measures.

Always appear as limits of compact cases.

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General formulation of the problem

$L(f) = \sum_{ij} g^{ij}\partial^2_{ij} f + b^i(x)\partial_i f$

b^i polynomials of degree ≤ 1 and g^{ij} polynomials of degree ≤ 2.

$b^i(x) = \sum_j \partial_j g^{ij}(x) + \sum_j g^{ij}\partial_j \log(\rho)$.

$\partial_i \log(\rho) = \sum_j (g^{-1})^{ij}\hat{L}_j$.

In addition : symmetry holds for every pair of polynomials

$\forall i,\ \sum_j g^{ij} n_j \rho(x) = 0$ on $\partial\Omega$, with $n_j =$ normal vector on $\partial\Omega$.

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General Formulation continued

$\partial\Omega$ is in some algebraic surface with degree $\leq 2n$.

With $\{D = 0\}$ the irreducible equation of $\partial\Omega$
$\forall i, \; \sum_j g^{ij}\partial_j D = 0$ on $\{D = 0\}$.

$\forall i, \; \sum_j g^{ij}\partial_j D = L_i D$ for some first order polynomials L_i.

The admissible domains are exactly those for which the above equation admits a non trivial solution.

If D has degree $2n$ (maximal), then it is proportional to the determinant of the metric. Then, the associated Laplace operator is a solution, corresponding to the measure $\rho(x) = D^{-1/2}$.

In general, D divides the determinant of the metric.

When such a D is a solution, with $D = D_1 \cdots D_k$ (irreducible factors) then any $\rho = D_1^{a_1} \cdots D_k^{a_k}$ is a solution for ρ.

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$\forall i,\ \sum_j g^{ij}\partial_j D = L_i D$ for some first order polynomials L_i.

Implies that $\{D = 0\}$ has no flex points and no flat points (in the complex projective 2-plane).

Implies that the dual curve has no singular points of some type

Leads through the above classification through the inspection of singular points of $\{D = 0\}$

(use Plucker formulas and classification through the genus of the associated Riemann surface).

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Genus method not available in higher dimension.

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Easy to construct models in 3-d from models in 2-d by double cover (pass from equation $P(x,y) = 0$ to equation $z^2 - P(x,y) = 0$) : works as soon as no cusp ad no double tangents.

Not able to show that every model should come from Lie group representation. Not even proved in the above double covers.

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In the 2-d case, many geometric models found when parameters of the measure are half-integers (similar to the Jacobi case)

Still a lot of geometric situations to identify.

Larger dimension

Genus method not available in higher dimension.

Not even able to prove that in the maximal degree case, the curvature is constant. Is that even true ?

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Thank You For Your Attention