# On the relaxation to equilibrium of random surfaces

Pietro Caputo
Havana, Cuba - March 8, 2012

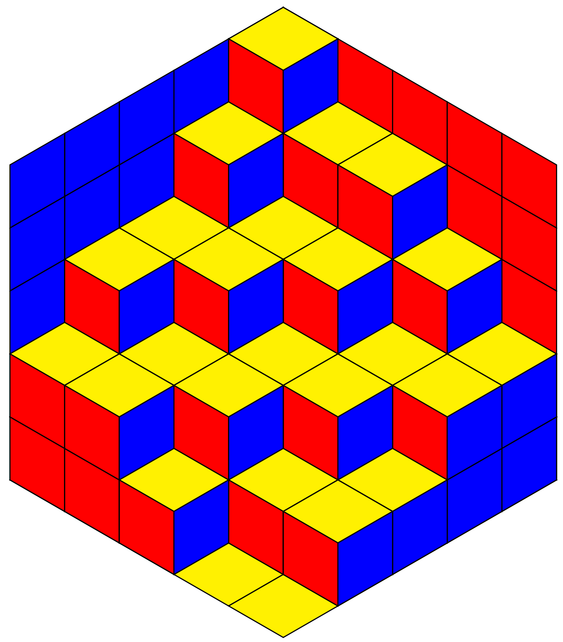
cf. P.C., F. Martinelli, F.Toninelli
*Comm. Math. Phys.* - to appear
and
P.C., F. Martinelli, F. Simenhaus, F.Toninelli
*Comm. Pure Appl. Math. 2011*

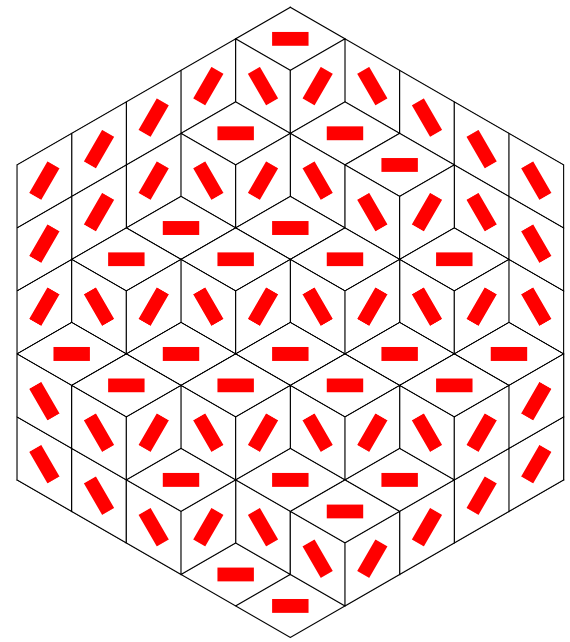
# Overview

- Two models of random surfaces
- Combinatorial structures: lozenge tilings, perfect matchings of honeycomb lattice
- Statistical physics: crystal shape, Ising model interfaces
- Dynamics: sampling via local Markov chains
- Goal: a unified approach for sharp mixing time bounds.
- Key new input: mean curvature motion as a driving mechanism behind equilibration.

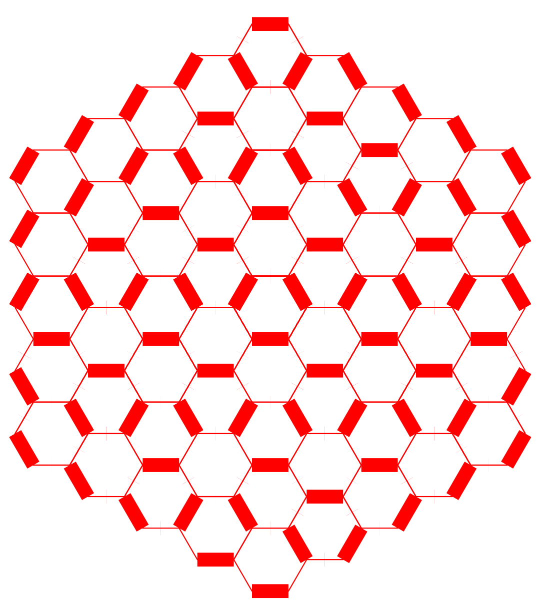
# Monotone surfaces, lozenge tilings, dimer coverings, Ising interface at T=0

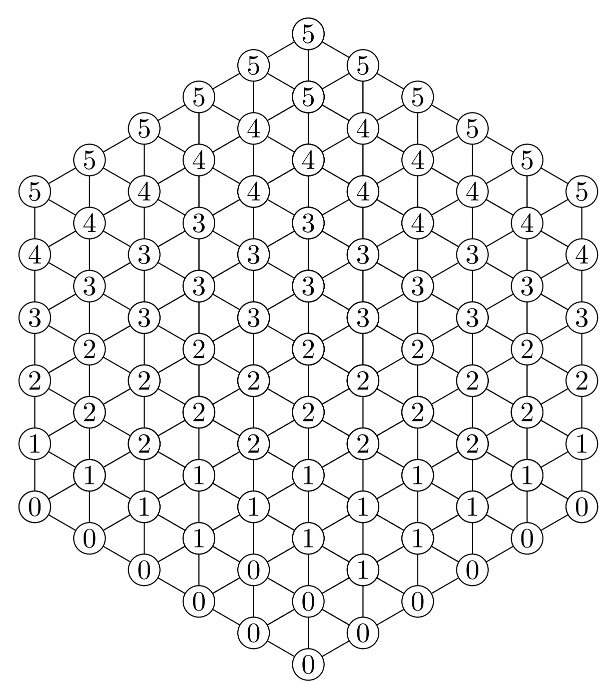
monotone
surface
or plane partition





perfect
matching
with
dimers





height
function

# Dynamics of monotone surfaces

$\{\phi_x\}_{x \in U \cup \partial U}$ , $\phi_x \in \mathbb{Z}$ , $\phi_x \geq \phi_y$ if $x_i \leq y_i$, $i = 1, 2$

Boundary condition: $\{\phi_x\}_{x \in \partial U}$ size : $L = \mathrm{diam}(U)$

## Continuous-time Markov chain:

indep. with rate one each column $x$ attempts $\phi_x \to \phi_x \pm 1$ with probab. $1/2$

move is accepted if compatible with constraints

equilibrium is uniform distribution $\pi$

**mixing time**: $T_{\mathrm{mix}} = \max_{\phi} \inf \left\{ t > 0 : \ \|\mu_t^{\phi} - \pi\| \leq 1/4 \right\}$

# Mixing time bounds

Conjectured: diffusive scaling $T_{\rm mix} = O(L^2 \log L)$

1. Luby-Randall-Sinclair (2000): Poly(L) bound via analysis of a column dynamics (non-local)

2. D.B. Wilson (2004): Sharp bounds for column dynamics, (approx. eigenfunction) $T_{\rm mix}^{CD} = O(L^2 \log L)$

3. Randall-Tetali (2002): Refined Poly(L) bound via comparison inequalities $T_{\rm mix} = \widetilde{O}(L^6)$

4. Further refinement via comparison and censoring (Peres-Winkler inequality): $T_{\rm mix} = \widetilde{O}(L^4)$

Notation: $\widetilde{O}(L^p)$ stands for $O(L^p (\log L)^c)$

# On boundary conditions

Wilson's technique + comparison methods are quite robust and can handle any boundary conditions

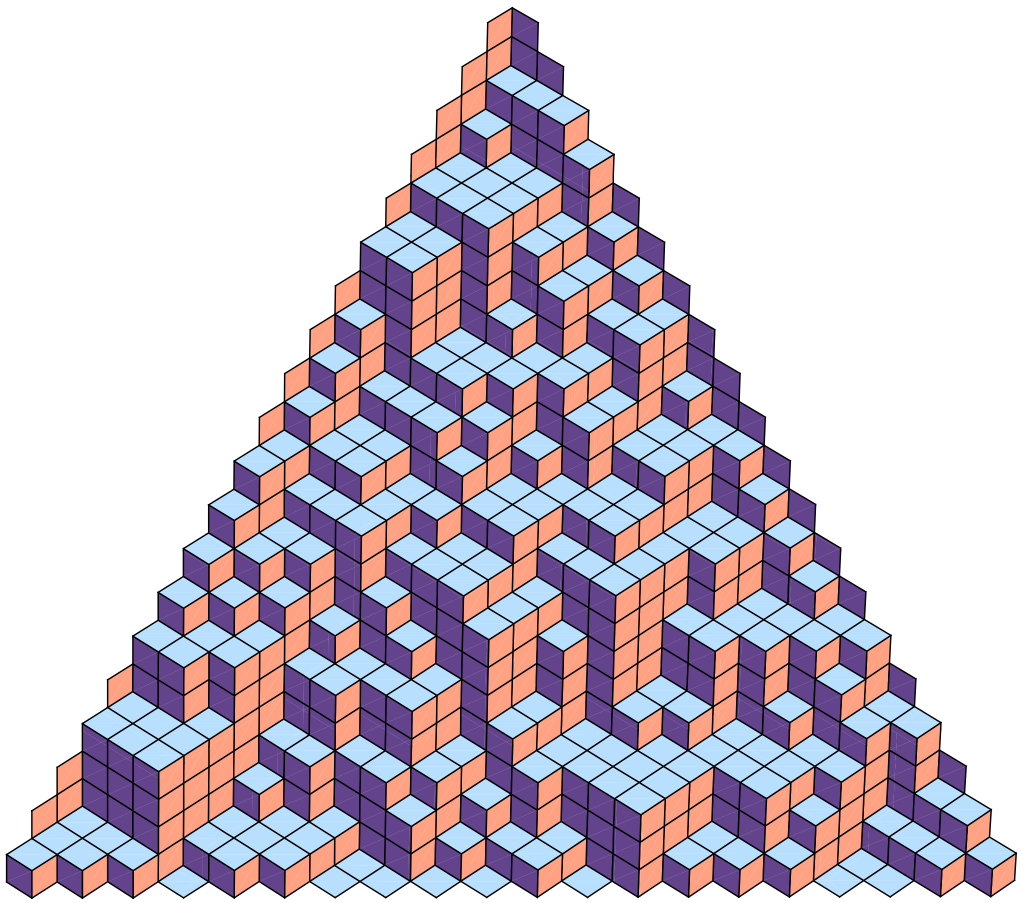
Our setting:
assume that the boundary heights lie (approx) on a given plane

Definition planar b.c.

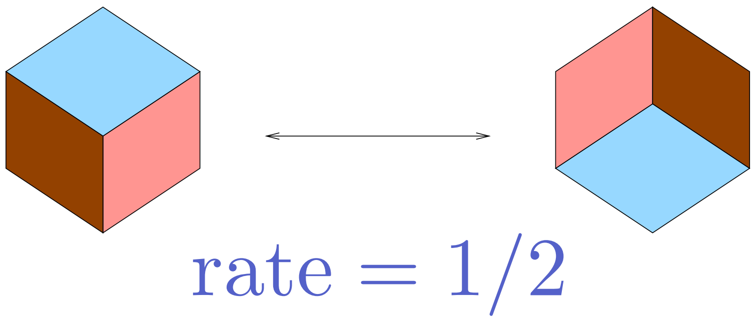
$$\exists \mathbf{n} : \; |\phi_x - \bar{\phi}(\mathbf{n})| \le C \log(1 + |x|)$$

where $\bar{\phi}(\mathbf{n})$ is the plane orthogonal to $\mathbf{n}$

# Monotone surfaces with planar boundary conditions

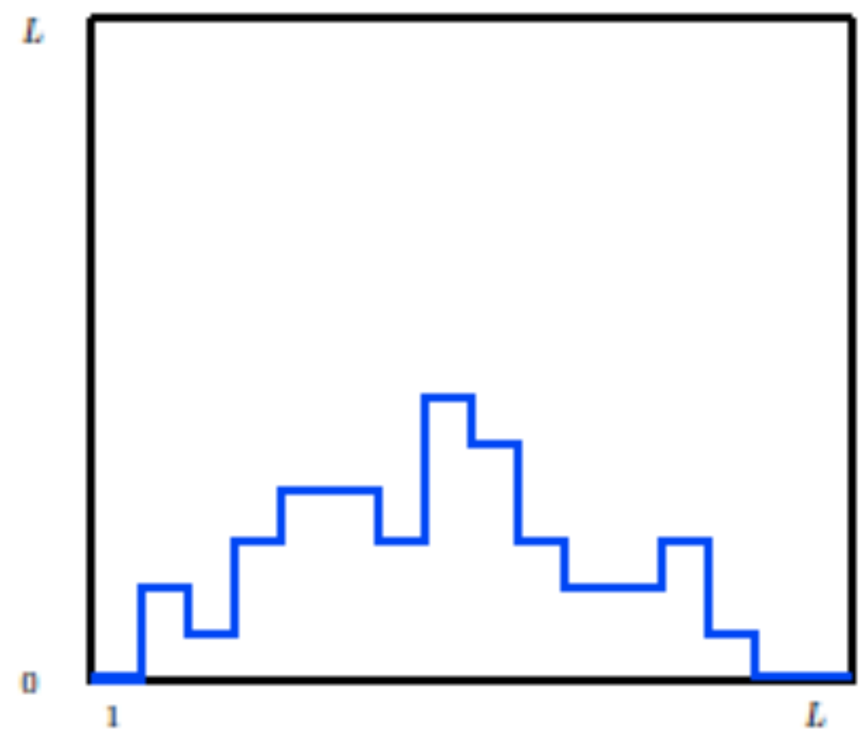


Corresponds to Ising, $T = 0$ with Dobrushin b.c. which impose an interface between $+$ and $-$ spins.




**Theorem :** $L^2/(c \log L) \le T_{\text{mix}} \le L^2 (\log L)^c$

# SOS model




$$\phi_i \in \{0, 1, \ldots, L\},\ i = 0, 1, \ldots L + 1$$

Gibbs measure $$\pi(\phi) := \frac{1}{Z} \exp(-\beta \sum_i |\phi_{i+1} - \phi_i|)$$

with boundary condition $$\phi_0 = \phi_{L+1} = 0$$

**Gibbs sampler**: with rate one each site moves up/down by one with probab. such that $\pi$ is reversible

**Conjecture** : $T_{\rm mix} = O(L^2 \log L)$

Remark: if $|\phi_{i+1} - \phi_i| \longrightarrow V(\phi_{i+1} - \phi_i)$

with $V''(x) \geq c > 0 \ \forall x$ (unif convex)

$\pi$ would satisfy Poincare' and log-Sobolev ineq. with $O(L^2)$

**Sinclair, Martinelli (2010)**: $T_{\rm mix} = \widetilde{O}(L^{2.5})$

**Theorem** : $T_{\rm mix} = \widetilde{O}(L^2)$

Proof by the same method used for monotone surfaces.

# Model features

## I. Equilibrium is macroscopically flat:

$$\Delta_L := \inf\{\Delta : \ \pi\left(\exists x : \ |\phi_x - \bar{\phi}_x| \geq \Delta\right) \leq L^{-100}\}$$

$\bar{\phi}$ is the average profile

SOS $\Delta_L = \widetilde{O}(L^{1/2})$ by simple random walk estimates

Mon Surf Theorem [CMST 2011] : $\Delta_L = \widetilde{O}(1)$

[non-trivial consequence of deep results in Kenyon, Okounkov, Sheffield *Ann.Math2006*]

## II. Monotonicity

The dynamical evolution preserves natural partial order on configurations - both initial and boundary conditions (monotone grand coupling)

# High level description of the proof

## Step I (model dependent)

If initial surface is within $\Delta_L$ from flat profile $\bar{\phi}$ then

$$||\mu_T^{\phi} - \pi|| \ll 1 \quad \forall T \geq \widetilde{O}(L^2)$$

## Step II (general strategy)

Flattening: surface reaches distance $\Delta_L$ from $\bar{\phi}$
in time $T = \widetilde{O}(L^2)$

Heuristic: motion by mean curvature

# Proof of Part I

Mon Surf: use Wilson's method for a restricted dynamics.

Idea: starting within $\Delta_L$ from $\bar{\phi}$ and using monotonicity one has that w.h.p. for all $t \leq L^{10}$

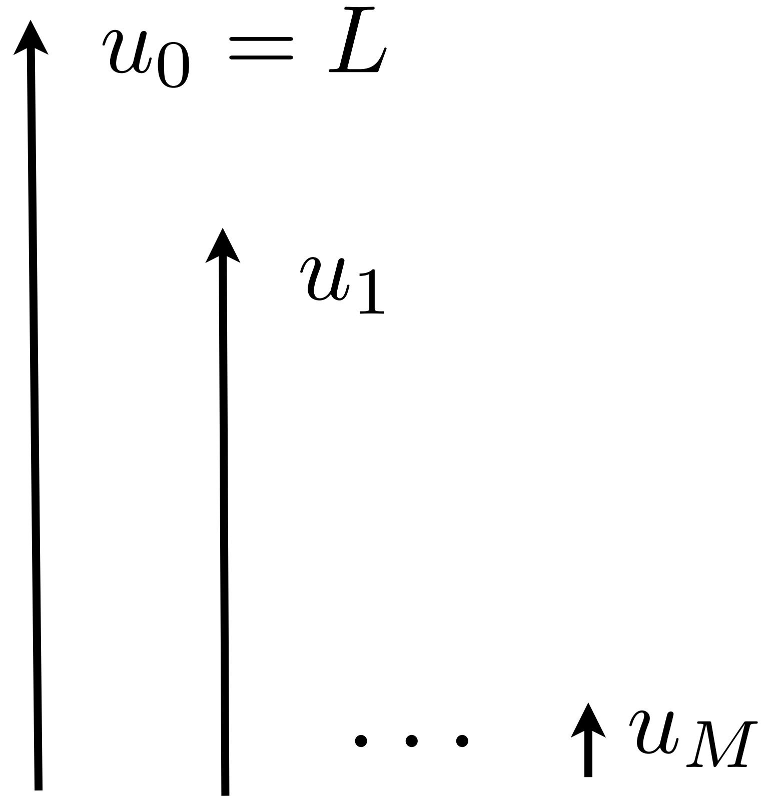
$$|\phi(t) - \bar{\phi}| \leq 2\Delta_L$$

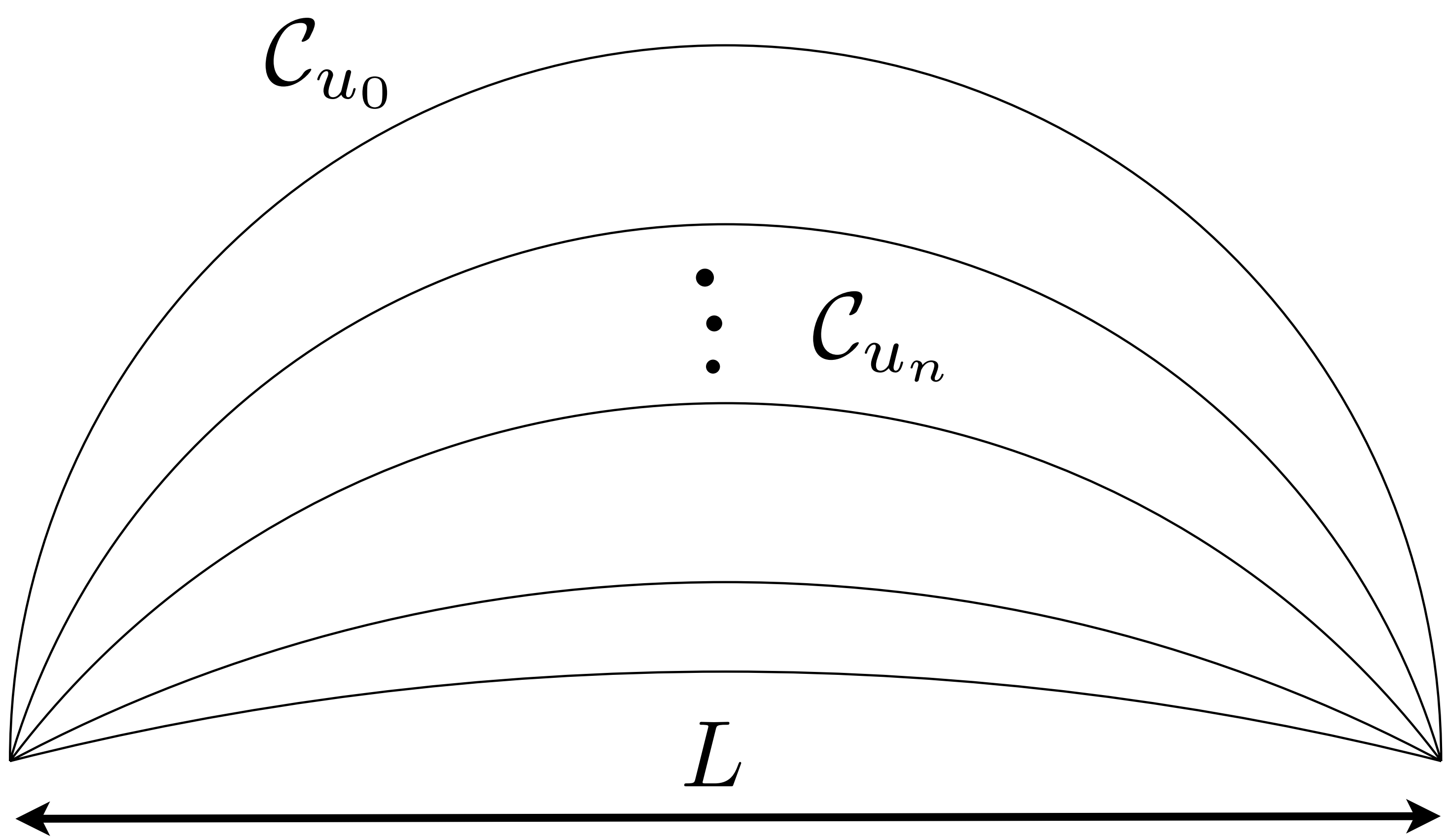
Remark: overhead between the mixing times in the comparison of local / non-local chain is proportional to $\max_i |\nabla \phi(i)|^2$

For SOS model: see Sinclair-Martinelli (2010) for smoothening [max gradients are small in equilibrium]

# Proof of Part II

Idea: start with maximal configuration and show that w.h.p. the dynamics is dominated by *deterministic* evolution




**want:** $u_M \sim \Delta_L$

$$\phi(t_n) \le \mathcal{C}_{u_n}\,, \quad n = 0, \dots, M \text{ with } t_0 = 0 \dots, t_M = \widetilde{O}(L^2)$$

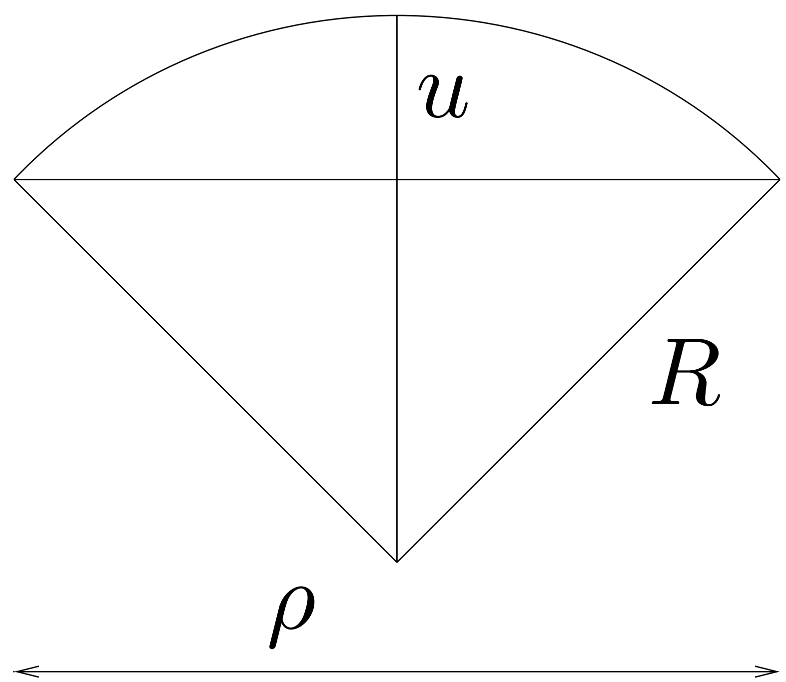
# Main Estimate

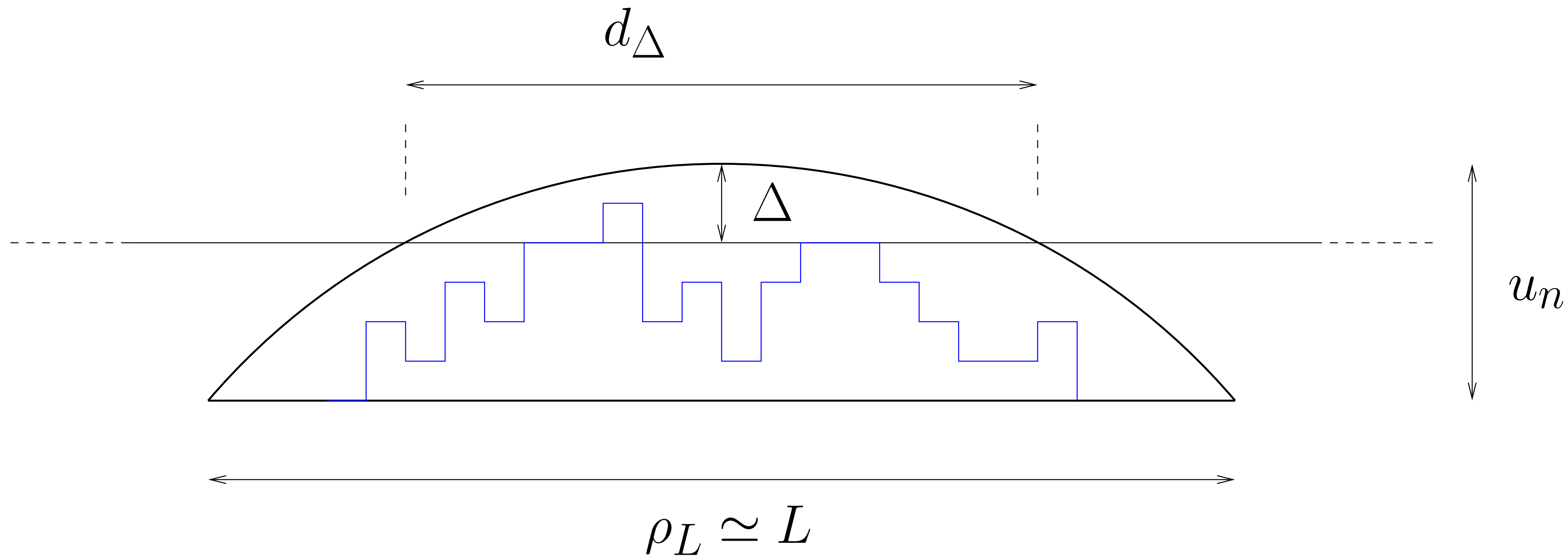
**Theorem**: There exists sequences u_n, t_n as above such that for all n< M and all times in $[t_n, L^{50}]$, w.h.p. $\phi(t) \le \mathcal{C}_{u_n}$

Radius of curvature $\quad (\text{if } u \ll R)$

$$(R-u)^2 + (\rho/2)^2 = R^2 \quad \Rightarrow \quad R(u) \sim \rho^2/8u$$

mean curvature: inward drift prop. to $1/R(u)$






$u_n - u_{n+1}$ = critical value of $\Delta$ such that eq. fluctuations on scale $d_\Delta$ are of order $\Delta$

Eq. fluctuations : $\widetilde{O}(\sqrt{d_\Delta})$

$$\rho^2/u_n \sim R(u_n) \sim d_\Delta^2/\Delta \sim d_\Delta^{3/2} \sim \Delta^3$$

Hence $\quad u_n - u_{n-1} \sim \Delta \sim (\rho^2/u_n)^{1/3} \sim R^{1/3}$

Times: $\quad t_{n+1} - t_n \sim d_\Delta^2 \sim (\rho^2/u_n)^{4/3} \sim R^{4/3}$

from Step 1: after this time surface has decreased by $\Delta$

$$u_n - u_{n-1} \sim (\rho^2/u_n)^{1/3}$$

$$\Rightarrow\ u_n^{4/3} - u_{n+1}^{4/3} \sim u_n^{1/3}(\rho^2/u_n)^{1/3} \sim \rho^{2/3}$$

$$\Rightarrow\ u_n^{4/3} \sim u_0^{4/3} - \rho^{2/3} n \ \sim L^{2/3}[L^{2/3} - n]$$

Thus $u_M \sim \Delta_L \sim \sqrt{L}$ requires $M \sim L^{2/3}$ steps

Moreover,

$$\sum_n (t_n - t_{n-1}) \sim \sum_{n=1}^{M} (\rho^2/u_n)^{4/3}$$

$$\sim L^{8/3} \sum_{n=1}^{M} u_n{}^{-4/3} \sim L^2 \sum_{n=1}^{M} (L^{2/3} - n)^{-1} = \widetilde{O}(L^2)$$

Remark: same recursion works for different scaling of max height fluctuations

$$\Delta_L = \widetilde{O}(L^\gamma),\ \gamma \in [0,1)$$

if maximal index M is such that $u_M = \Delta_L$

remarkably: $t_M = \widetilde{O}(L^2)$

For monotone surfaces: $\gamma = 0$

same recursion but new scales:

$$u_n - u_{n+1} = 1,\ t_{n+1} - t_n = \widetilde{O}(R(u_n))$$

$$u_M = (\log L)^a,\ t_M = \widetilde{O}(L^2)$$

# Open problems:

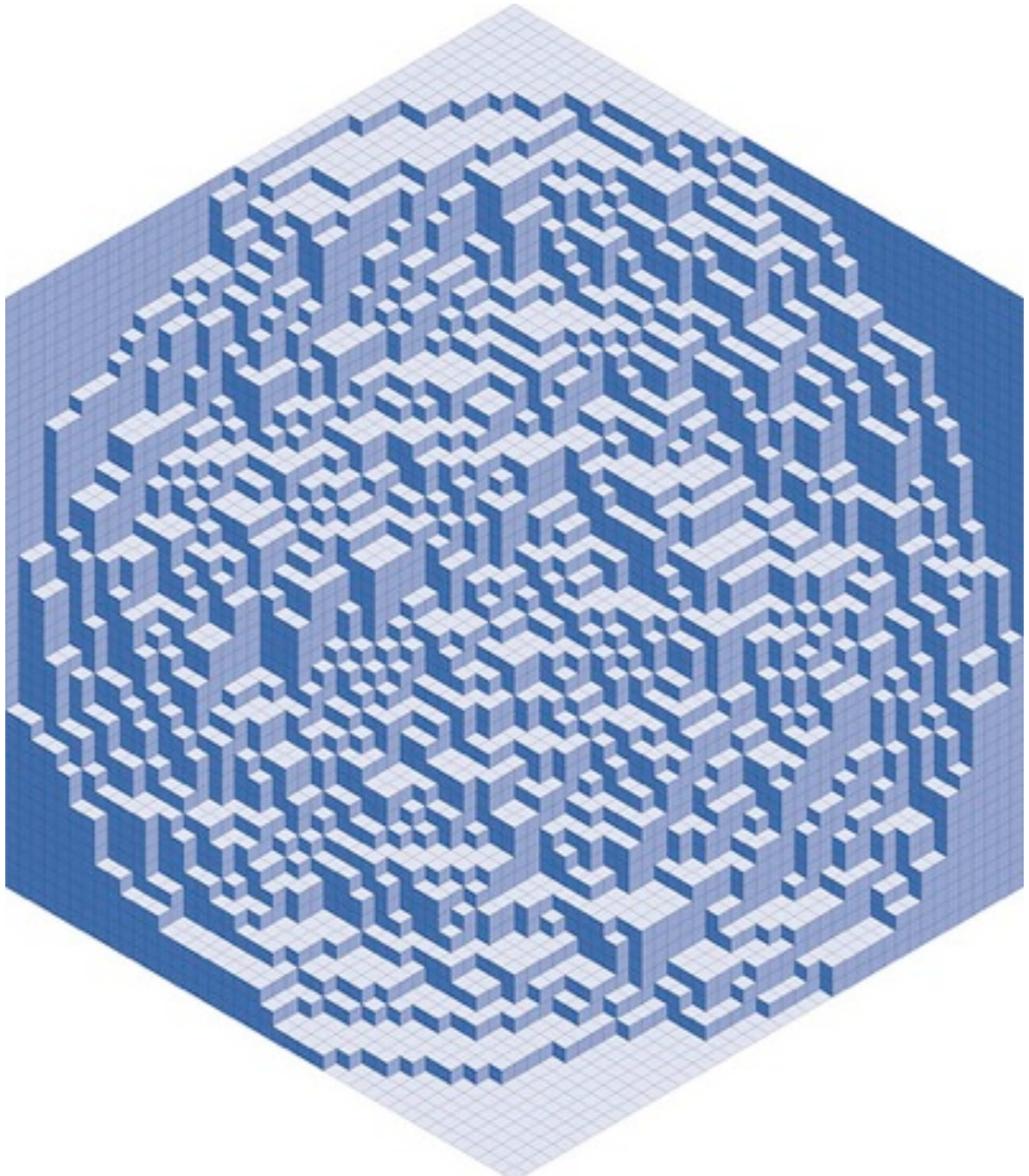
I. Removal of spurious log L powers: establish $T_{\text{mix}} \asymp L^2 \log L$

or spectral gap estimate $\text{gap} \asymp L^{-2}$

[would allow hydrodynamic limit, cf. Funaki's work]

II. Extension to non-planar boundary conditions:

Relaxation to limiting shape ?



# Non-planar case: main difficulties

In the limit the surface is close to a non-flat limiting shape.

Arctic circle phenomena [Cohn, Larsen, Propp, Kenyon...]

Equilibrium results are far from what is needed here.

Questions:

(i) how close ?

(ii) for finite large L: order of max fluctuations ?

(iii) sharp control of the limiting shape as a function of the underlying domain ?

# 3D stochastic Ising model at zero-temperature

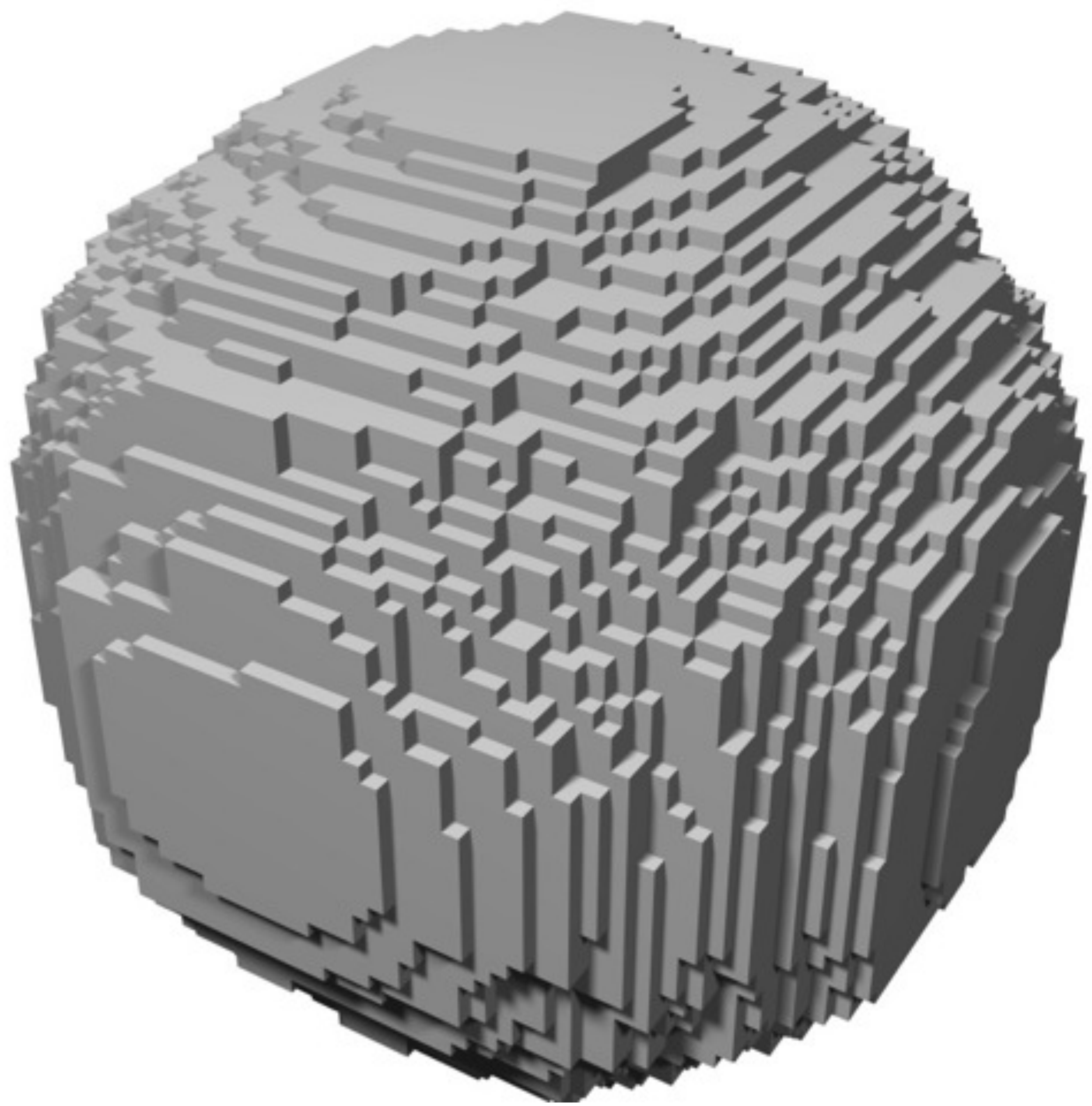
At each $i \in \mathbb{Z}^3$ there is a spin $\sigma_i = \pm 1$ and an i.i.d. Poisson clock of mean $1$.

When the clock labeled $i$ rings, update $\sigma_i$ as follows: if three neighbors are $+$ and three are $-$, set $\sigma_i = \pm$ with equal probabilities; otherwise, set $\sigma_i$ equal to the majority of its neighbors.

Initial condition: $\sigma_i(t=0) = -$ if $i \in \{-L, \ldots, L\}^3$ and $\sigma_i(t=0) = +$ otherwise.

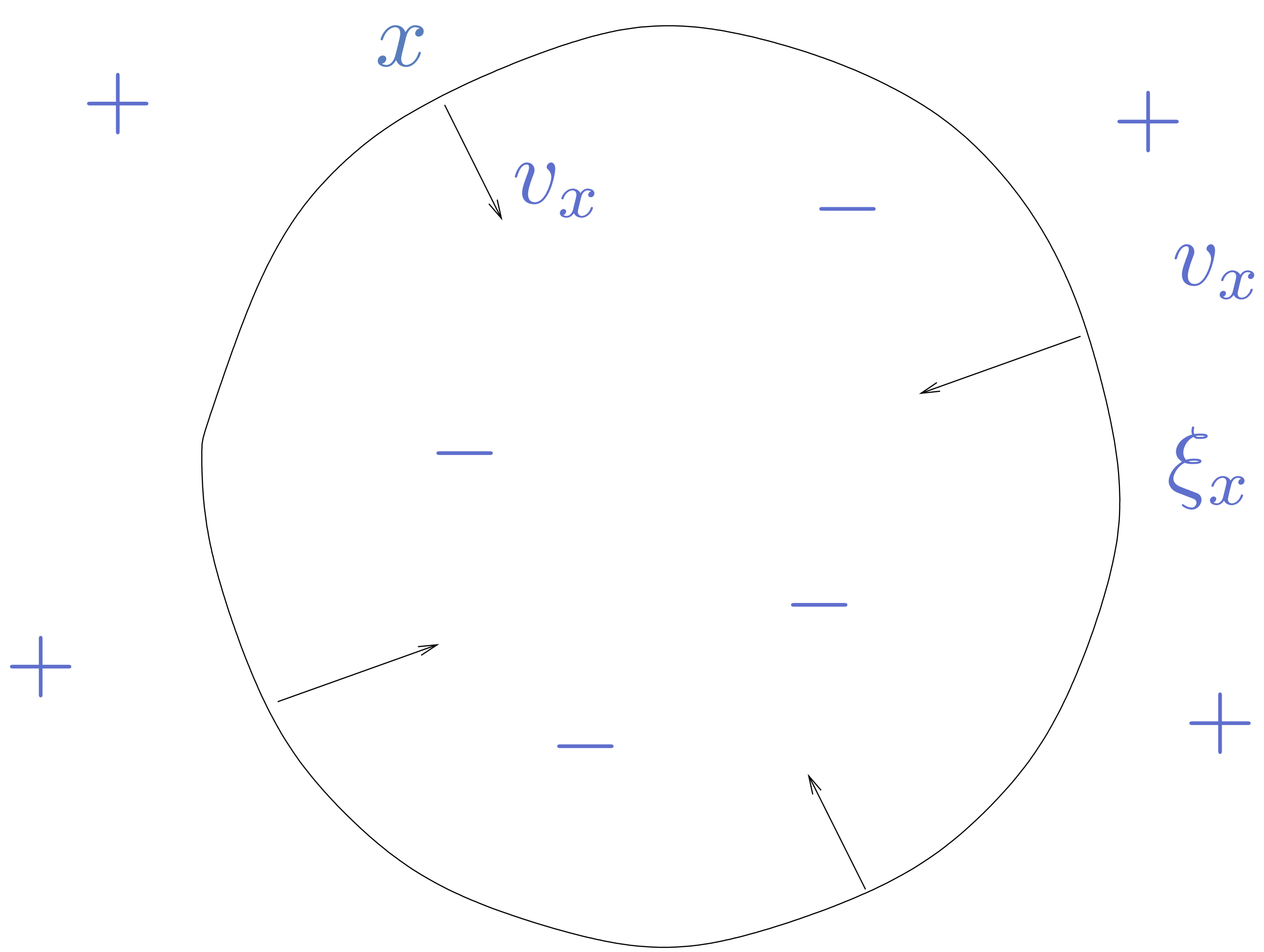
Let $\tau_+ = \inf\{t > 0 : \sigma_i(t) = + \ \forall i \in \mathbb{Z}^3\}$

# A "spherical" droplet of minus spins (grey) in a sea of pluses



# Conjectured behavior of $\tau_+$

One expects $\tau_+ \sim cL^2$ (actually, in every dimension $d$ ).




$$v_x = -\vec{n}_x c(\vec{n}_x)\, \xi_x$$

$\xi_x$ : mean curvature

Heuristics: anisotropic motion by mean curvature of the -1 domain.

# The two-dimensional case

For the problem on $\mathbb{Z}^2$ one can prove that

$$\mathbf{P}(\tau_+ \geq cL^2) \leq e^{-cL}$$

(Fontes, Schonmann, Sidoravicius CMP '02)
and

$$\mathbf{P}(\tau_+ \leq (1/c)L^2) \leq e^{-cL}$$

(C-Martinelli-Simenhaus-Toninelli CPAM '11)

for a suitable constant $c$, where $\mathbf{P}$ is the law of the Glauber dynamics

[comparison with simple exclusion,
detailed analysis of the inward drift]

# Back to 3D: trivial bounds

Easy to prove: with high probability,

$$(1/c)L \leq \tau_+ \leq cL^3$$

for a suitable constant $c$

Lower bound: in unit time, volume decreases at most by $L^2$ (surface area of the boundary between $+$ and $-$)

Upper bound: easy comparison with 2D

# Main Result

Theorem [CMST] There exists $0 < c < \infty$ such that

$$\mathbf{P}\left(\frac{L^2}{c\log L} \le \tau_+ \le L^2(\log L)^c\right) \xrightarrow{L\to\infty} 1$$

Lower bound: related to a question on ordered random walks

Upper bound: as before, dynamics and equilibrium fluctuations of discrete monotone interfaces, domination by deterministic local flattening

dimension $d \ge 4$ [H.Lacoin]. Lower bound is open.