An estimate of the effect of nonlinear term in the stochastic Navier-Stokes equation characterizing the "energy cascade" in a turbulent flow

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1 The Kolmogorov-Obukhov's idea

This paper draws inspiration from the theory of Kolmogorov and Obukhov presented in 1941 [4].

The paper of Kolmogorov and Obukhov has the aim to construct the **turbulence spectral theory** and it consists in the study of energy distribution among perturbations of different length in a steady state (*theory K41*).

The Kolmogorov and Obukhov's theory bases on the physical model that supposes the velocity field of a turbulent flow is characterized by perturbations of different lengths which exercise qualitatively different influences on a turbulent phenomenon.

The idea of Kolmogorov and Obukhov is:

- to separate the fluid motion in “macrocomponent”, that consists of “big” perturbations compared to a choosen observation scale, characterized by a low frequency, and “microcomponent”, that is “small” perturbations, with high frequency

- to consider Navier-Stokes equations system related to the macrocomponent and the one related to the microcomponent separately

- using mathematical expectation, to show that there is an energy transfer from macrocomponent to microcomponent.

The cause of such an energy transfer could be ascribed to the presence of the **nonlinear term**.

The purpose of this paper is to give an estimate of the eventual energy transfer from motions of big length (local average) to motions of small length (fluctuations) in the stationary solution.

2 Stochastic Navier-Stokes equations

We consider the simpler $2D$ case.

The approach presented by Kolmogorov and Obukhov [4] to study the turbulence problem is a probabilistic approach. To describe the turbulent flow of an incompressible viscous fluid, we introduce the stochastic Navier-Stokes equations system

$$dv = [-(v \cdot \nabla)v + \nu \Delta v - \nabla p]dt + dW \tag{2.1}$$

$$\nabla \cdot v = 0$$

$v(t,x) = (v_1(t,x), v_2(t,x))$: the velocity
$p(t,x)$: the pressure
ν: the viscosity coefficient
$W(t)$: the Brownian motion.

Here we consider $W(t)$ as a Brownian motion with values in a suitable Hilbert space.

Domain

Because of the complexity of the problem, in this paper we consider the stochastic Navier-Stokes equation system on the bi-dimensional torus

$$\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2,$$

that corresponds to consider *periodic functions* of period 2π both in x_1 and in x_2 and it allows us to make use of the Fourier series.

Functional spaces

Setting

$$V^\infty = \{v \in C^\infty(\mathbb{T}^2;\mathbb{R}^2) \,|\, \nabla \cdot v = 0, \int_{\mathbb{T}^2} v dx = 0\} \tag{2.2}$$

we define

$$\mathcal{H} = \overline{V^\infty}^{L^2(\mathbb{T}^2;\mathbb{R}^2)},$$

$$\langle u, v\rangle_{\mathcal{H}} = \int_{\mathbb{T}^2} u \cdot v dx,$$

from which follows that $(\mathcal{H}, ||.||_{\mathcal{H}})$ is an Hilbert space.

We introduce the orthogonal projection operator $\mathcal{P}_{\mathcal{H}}$ from $L^2(\mathbb{T}^2;\mathbb{R}^2)$ on $\mathcal{H}$. Thanks to the choice of the domain, the following properties hold:

$$\mathcal{P}_{\mathcal{H}}\nabla p = 0 \tag{2.3}$$

$$\mathcal{P}_{\mathcal{H}}\Delta v = \Delta v.$$

The system (2.1) can be written in the form

$$dv = [-\mathcal{P}_{\mathcal{H}}(v \cdot \nabla)v + \nu\Delta v]dt + dW. \tag{2.4}$$

Observation: if $v \in \mathcal{H}$, then there exists a periodic scalar function ψ such that

$$v = (v_1, v_2), \qquad v_1 = -\frac{\partial\psi}{\partial x_2}, \quad v_2 = \frac{\partial\psi}{\partial x_1}. \tag{2.5}$$

As ψ is a periodic function of period 2π both in x_1 and in x_2, it can be developed using *Fourier series.*

Fourier series: preliminaries

We set

$$\mathbb{Z}^2_1 = \{\, k = (k_1, k_2) \in \mathbb{Z}^2 \,|\, k_1 > 0 \vee (k_1 = 0, k_2 > 0)\,\}. \tag{2.6}$$

The functions

$$\frac{1}{\sqrt{2}\pi}\cos(k \cdot x), \qquad \frac{1}{\sqrt{2}\pi}\sin(k \cdot x), \qquad k \in \mathbb{Z}^2_1 \tag{2.7}$$

form an orthonormal basis of $L^2(\mathbb{T}^2;\mathbb{R})$.

Then $\psi \in L^2(\mathbb{T}^2;\mathbb{R})$ can be expressed in Fourier series

$$\psi(t,x) = \sum_{k\in\mathbb{Z}^2_1} \big(\tilde{\alpha}_k \frac{1}{\sqrt{2}\pi}\cos(k \cdot x) + \tilde{\beta}_k \frac{1}{\sqrt{2}\pi}\sin(k \cdot x)\big) + \frac{\tilde{\alpha}_0}{2\pi} \tag{2.8}$$

where

$$\tilde{\alpha}_k = \int_{\mathbb{T}^2} \psi \frac{1}{\sqrt{2}\pi} \cos(k\cdot x)dx, \quad \tilde{\beta}_k = \int_{\mathbb{T}^2} \psi \frac{1}{\sqrt{2}\pi} \sin(k\cdot x)dx, \;\; k \in \mathbb{Z}_1^2$$

and

$$\tilde{\alpha}_0 = \int_{\mathbb{T}^2} \frac{\psi}{2\pi} dx.$$

It follows that $v \in \mathcal{H}$ has the form

$$v(t,x) = -\sum_{k\in\mathbb{Z}_1^2} k^\perp \big(\tilde{\alpha}_k \frac{\sin(k\cdot x)}{\sqrt{2}\pi} - \tilde{\beta}_k \frac{\cos(k\cdot x)}{\sqrt{2}\pi}\big), \quad k^\perp = (-k_2, k_1). \tag{2.9}$$

Description of the stochastic Navier-Stokes equation by means of Fourier series.

We set

$$e_{1,k} = -\frac{k^\perp}{|k|\sqrt{2}\pi} \sin(k\cdot x), \tag{2.10}$$

$$e_{2,k} = \frac{k^\perp}{|k|\sqrt{2}\pi} \cos(k\cdot x). \quad |k| = \sqrt{k_1^2 + k_2^2}$$

and

$$\mathbb{L} = \{1,2\} \times \mathbb{Z}_1^2.$$

$\{e_{j,k}\}_{(j,k)\in\mathbb{L}}$ form an orthonormal basis of $\mathcal{H}$.

Each $v \in \mathcal{H}$ has the representation

$$v = \sum_{(j,k)\in\mathbb{L}} \langle e_{j,k}, v\rangle e_{j,k} = \sum_{(j,k)\in\mathbb{L}} \alpha_{j,k} e_{j,k}, \tag{2.11}$$

with

$$\alpha_{1,k} = \langle e_{1,k}, v\rangle = -\int_{\mathbb{T}^2} \frac{k^\perp}{|k|} \frac{\sin(k\cdot x)}{\sqrt{2}\pi} \cdot v(x)dx,$$

$$\alpha_{2,k} = \langle e_{2,k}, v\rangle = \int_{\mathbb{T}^2} \frac{k^\perp}{|k|} \frac{\cos(k\cdot x)}{\sqrt{2}\pi} \cdot v(x)dx.$$

We consider a Brownian motion $W(t)$ with values in $\mathcal{H}$ expressed in the form

$$W(t) = \sum_{(j,k)\in\mathbb{L}} \lambda_{j,k} e_{j,k} W^{(j,k)}(t), \qquad \lambda_{j,k} \in \mathbb{R}, \tag{2.12}$$

where $W^{(j,k)}(t)$ are independent real valued Brownian motions.

The Fourier series representation of the **nonlinear term** is

$$\begin{aligned}
(v\cdot\nabla)v &= \big(v_1\frac{\partial}{\partial x_1}v_1 + v_2\frac{\partial}{\partial x_2}v_1, v_1\frac{\partial}{\partial x_1}v_2 + v_2\frac{\partial}{\partial x_2}v_2\big) = \\
&= \frac{1}{4}\sum_{k'\in\mathbb{Z}^2_*}\sum_{k''\mathbb{Z}^2_*}\frac{k''^{\perp}(k'^{\perp}\cdot k'')}{|k'|\,|k''|}\times \\
&\times \Big[\frac{\alpha_{1,k'}\alpha_{1,k''}-\alpha_{2,k'}\alpha_{2,k''}}{2(\sqrt{2}\pi)^2}\sin((k'+k'')\cdot x)+ \\
&+\frac{\alpha_{1,k'}\alpha_{1,k''}+\alpha_{2,k'}\alpha_{2,k''}}{2(\sqrt{2}\pi)^2}\sin((k'-k'')\cdot x)+ \\
&+\frac{\alpha_{1,k'}\alpha_{2,k''}-\alpha_{1,k''}\alpha_{2,k'}}{2(\sqrt{2}\pi)^2}\cos((k'-k'')\cdot x)+ \\
&-\frac{\alpha_{1,k'}\alpha_{2,k''}+\alpha_{1,k''}\alpha_{2,k'}}{2(\sqrt{2}\pi)^2}\cos((k'+k'')\cdot x)\Big].
\end{aligned} \tag{2.13}$$

with $k \in \mathbb{Z}^2_* = \mathbb{Z}^2\backslash\{(0,0)\}$.

Observation: The expression (2.13) drops us a hint that the presence of the nonlinear term can be responsible for a transfer of momentum from waves of some length to waves of different length.

Observation: The $2D$ term $(v\cdot\nabla)v$ has the properties

$$\langle (v\cdot\nabla)v, v\rangle = 0 \qquad \forall v \in \mathcal{H}, \tag{2.14}$$

$$\langle (v\cdot\nabla)v, \Delta v\rangle = 0 \qquad \forall v \in \mathcal{H}. \tag{2.15}$$

As Kolmogorov and Obukhov were interested in studying the energy transfer in the steady state, we will examine the energy balance when v realizes the invariant measure, that is the measure associated to the stationary solution.

3 Invariant measure

Definition

Let H be an Hilbert space, let μ be a measure defined on the Borel σ-algebra $\mathcal{B}(H)$ of H.

If there exists a stochastic process $X = X(w, t)$ with values in H such that

i. X is the solution of the stochastic equation

$$dX = F(X,t)dt + G(X,t)dW \tag{3.1}$$

ii. for all $t \geq 0$, the measure generated by the random variable $X(\cdot, t)$ coincides with μ, that is

$$\mathbb{P}(X(\cdot,t) \in B) = \mu(B) \quad \forall t \geq 0, \quad \forall B \in \mathcal{B}(H)$$

then the measure μ is called *invariant measure* for the stochastic equation (3.1).

With regard to the stochastic Navier-Stokes equation (2.4), the existence of an invariant measure has been proved (Cruzeiro 1989 [2], etc.)

With regard to the study of unicity, see for instance the papers of Albeverio and Ferrario (2004) [1] and of Da Prato and Debussche (2002) [3].

4 Kinetic energy balance

To examine the kinetic energy balance we apply the Ito's formula (see for instance [5]) to the function

$$\psi_1(t) = \frac{1}{2} \|v\|^2_{L^2(\mathbb{T}^2)}$$

and, thanks to $\langle (v\cdot\nabla)v, v\rangle = 0$, we obtain

$$\mathbb{E}\psi_1(t) - \mathbb{E}\psi_1(0) = -\nu\int_0^t \mathbb{E}\,\|\nabla v\|^2_{L^2(\mathbb{T}^2)}dt' + \frac{1}{2}\sum_{(j,k)\in\mathbb{L}}\lambda^2_{j,k}t. \tag{4.1}$$

In particular, if v is the process that realizes the invariant measure, then we have

$$\nu\mathbb{E}\,\|\nabla v\|^2_{L^2(\mathbb{T}^2)} = \frac{1}{2}\sum_{(j,k)\in\mathbb{L}}\lambda^2_{j,k}. \tag{4.2}$$

The equality (4.2) shows that the energy due to the perturbation is equal to the one dissipated by viscosity.

The Fourier series expression of (4.2) is

$$\nu\sum_{(j,k)\in\mathbb{L}}|k|^2\,\mathbb{E}\alpha^2_{j,k} = \frac{1}{2}\sum_{(j,k)\in\mathbb{L}}\lambda^2_{j,k}. \tag{4.3}$$

The k-th element expression of (4.3) is

$$\mathbb{E}\big[\alpha_{j,k}\,\langle e_{j,k}, (v\cdot\nabla)v\rangle\,\big] + \nu\,|k|^2\,\mathbb{E}\alpha^2_{j,k} = \frac{1}{2}\lambda^2_{j,k}. \tag{4.4}$$

Observation: from the expressions (4.3) and (4.4) we can observe the presence of the nonlinear term on the k-th element and the sum of all the k-th contributions of the nonlinear term is equal to zero.

To define the macrocomponent and to study its energy balance, we introduce a particular local average operator.

5 Local average

Let $\varphi(x)$ be a function defined on $\mathbb{R}^d$.
We consider a function $\tilde{\Theta}(x)$ such that

$$\tilde{\Theta}(x) \geq 0, \qquad \int_{\mathbb{R}^d}\tilde{\Theta}(x)dx = 1$$

(and, if necessary, with some other suitable conditions).

We define the *local average* by means of the convolution operator $\tilde{\Theta}*$.

The local average of φ:

$$\overline{\varphi}(x) = (\tilde{\Theta} * \varphi)(x) = \int_{\mathbb{R}^d} \tilde{\Theta}(x-y)\varphi(y)dy. \tag{5.1}$$

In this paper we choose the family of functions

$$\Theta_\delta(x) = \mathcal{F}^{-1}\Big(\frac{1}{1+\delta|\xi|^2}\Big)(x), \qquad \delta > 0, \tag{5.2}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform and δ is the characteristic width of the weight function that we use to define the local average.
We have $\Theta_\delta * = (1-\delta\Delta)^{-1}$.

A so defined function $\Theta_\delta(x)$ presents the following $2D$ explicit form

$$\Theta_\delta(x) = \frac{1}{2\pi}\int_0^{+\infty} \frac{\exp(-|x|\sqrt{\frac{1}{\delta}+\xi_2^2})}{\delta\sqrt{\frac{1}{\delta}+\xi_2^2}} d\xi_2 \tag{5.3}$$

(in the proof we use the residue theorem).

Observing that

$$\|\Theta_\delta * \varphi\|_{L^\infty(\mathbb{R}^2)} \leq \|\Theta_\delta\|_{L^1(\mathbb{R}^2)} \|\varphi\|_{L^\infty(\mathbb{R}^2)} = \|\varphi\|_{L^\infty(\mathbb{R}^2)}, \tag{5.4}$$

it is possible to define the local average operator $\Theta_\delta *$ over the set of functions $L^\infty(\mathbb{R}^2)$.

As $e_{j,k} \in L^\infty(\mathbb{R}^2)$, $\Theta_\delta * e_{j,k}$ is well defined and we have

$$\Theta_\delta * e_{j,k} = \frac{1}{1+\delta|k|^2} e_{j,k}, \qquad j = 1, 2, \quad k \in \mathbb{Z}_1^2. \tag{5.5}$$

Definition of the operator $\Theta_\delta *$ on $\mathcal{H}$.

Given $v \in \mathcal{H}$, from relation (5.5) we have

$$\Theta_\delta * v = \sum_{(j,k)\in\mathbb{L}} \langle e_{j,k}, v\rangle \Theta_\delta * e_{j,k} = \sum_{(j,k)\in\mathbb{L}} \frac{\langle e_{j,k}, v\rangle}{1+\delta\,|k|^2} e_{j,k}. \tag{5.6}$$

So for $v \in \mathcal{H}$ also it will be called $\Theta_\delta * v$ the local average of v and we set

$$\overline{v} = \Theta_\delta * v. \tag{5.7}$$

The difference

$$u = v - \overline{v} \tag{5.8}$$

can be called **fluctuation**, in conformity with the common use of terminology in the study of turbulence.

6 The macrocomponent energy balance

To examine the macrocomponent energy balance we apply Ito's formula to the function

$$\psi_2(t) = \frac{1}{2}\,\|\overline{v}\|^2_{L^2(\mathbb{T}^2)} = \frac{1}{2}\,\|\Theta_\delta * v\|^2_{L^2(\mathbb{T}^2)}$$

and we obtain

$$\begin{aligned} \mathbb{E}\psi_2(t) - \mathbb{E}\psi_2(0) = &-\int_0^t \mathbb{E}\,\langle \Theta_\delta * ((v\cdot\nabla)v), \overline{v}\rangle\, dt' \\ &- \nu \int_0^t \mathbb{E}\,\|\nabla\overline{v}\|^2_{L^2(\mathbb{T}^2)} dt' + \frac{1}{2}\sum_{(j,k)\in\mathbb{L}} \frac{1}{(1+\delta\,|k|^2)^2}\lambda^2_{j,k} t. \end{aligned} \tag{6.1}$$

In particular, if $v(t)$ is the process that realizes the invariant measure, then we have

$$\mathbb{E}\langle \Theta_\delta * \mathcal{P}_{\mathcal{H}}((v\cdot\nabla)v), \overline{v}\rangle + \nu\mathbb{E}\|\nabla\overline{v}\|^2_{L^2(\mathbb{T}^2)} = \frac{1}{2}\sum_{(j,k)\in\mathbb{L}} \frac{1}{(1+\delta\,|k|^2)^2}\lambda^2_{j,k}, \tag{6.2}$$

which Fourier series version is
(6.3)

$$\sum_{(j,k)\in\mathbb{L}} \mathbb{E}\Big[\frac{1}{(1+\delta\,|k|^2)^2}\alpha_{j,k}\,\langle e_{j,k},(v\cdot\nabla)v\rangle\Big] + \nu \sum_{(j,k)\in\mathbb{L}} \frac{|k|^2}{(1+\delta\,|k|^2)^2}\mathbb{E}\alpha_{j,k}^2$$
$$= \frac{1}{2}\sum_{(j,k)\in\mathbb{L}} \frac{1}{(1+\delta\,|k|^2)^2}\lambda_{j,k}^2.$$

Observation:
Comparing the total energy balance

$$\nu\mathbb{E}\,\|\nabla v\|^2_{L^2(\mathbb{T}^2)} = \frac{1}{2}\sum_{(j,k)\in\mathbb{L}} \lambda_{j,k}^2 \tag{6.4}$$

with the macrocomponent energy balance

$$\mathbb{E}\langle\Theta_\delta * \mathcal{P}_{\mathcal{H}}((v\cdot\nabla)v),\overline{v}\rangle + \nu\mathbb{E}\|\nabla\overline{v}\|^2_{L^2(\mathbb{T}^2)} = \frac{1}{2}\sum_{(j,k)\in\mathbb{L}} \frac{1}{(1+\delta\,|k|^2)^2}\lambda_{j,k}^2, \tag{6.5}$$

we observe that the nonlinear term

$$\mathbb{E}\langle\Theta_\delta * \mathcal{P}_{\mathcal{H}}((v\cdot\nabla)v),\overline{v}\rangle$$

can subtract energy from the mean field and it can be responsible for the energy transfer from mean local velocity to fluctuations.

The equality (6.5) gives the possibility to interpret the phenomenon of the energy “cascade” from local average motion $\overline{v} = \Theta_\delta * v$ to fluctuations $u = v - \overline{v}$ by means of the nonlinear term.

7 An estimate of the nonlinear term

Theorem: *We suppose that v is a solution of the Navier-Stokes equations*

$$dv = [-\mathcal{P}_{\mathcal{H}}(v\cdot\nabla)v + \nu\Delta v]dt + dW \tag{7.1}$$

realizing an invariant measure. Then there exists a function $f(\delta)$, $\delta > 0$, such that

$$f(\delta) \to 0 \qquad \text{as} \quad \delta \to 0$$

and that

$$\nu\mathbb{E}\|\nabla\overline{v}\|^2_{L^2(\mathbb{T}^2)} + N_\delta = \frac{1}{2}\sum_{(j,k)\in\mathbb{L}} \frac{1}{(1+\delta\,|k|^2)^2}\lambda^2_{j,k}, \tag{7.2}$$

$$\begin{aligned} |N_\delta| &\leq C_1 f(\delta)\mathbb{E}[\|v\|_{L^2}\,\|\nabla v\|_{L^2}\,\|\Delta v\|_{L^2}] \\ &\leq C_2 f(\delta)\sum_{(j,k)\in\mathbb{L}}\lambda^2_{j,k}\Big(\sum_{(j,k)\in\mathbb{L}}|k|^2\lambda^2_{j,k}\Big)^{\frac{1}{2}}, \end{aligned} \tag{7.3}$$

where

$$N_\delta = \sum_{(j,k)\in\mathbb{L}}\mathbb{E}\big[\frac{1}{(1+\delta\,|k|^2)^2}\alpha_{j,k}\,\langle e_{j,k},(v\cdot\nabla)v\rangle\big] \tag{7.4}$$

and with the constants C_1, C_2 *independent of* δ *and of* $\{\lambda_{j,k}\}_{(j,k)\in\mathbb{L}}$. *More precisely, if* $\mathbb{Z}^2_* = \mathbb{Z}^2\backslash\{(0,0)\}$, *the function* $f(\delta)$ *is given by*

$$f(\delta) = \inf_{0<\varepsilon<1}\Big[\Big(\sum_{k'\in\mathbb{Z}^2_*}\frac{1}{|k'|^{2(1+\varepsilon)}}\Big)^{\frac{1}{2}}\sup_{k\in\mathbb{Z}^2_*}\frac{1}{|k|^{(1-\varepsilon)}}\frac{2\delta\,|k|^2+\delta^2\,|k|^4}{(1+\delta\,|k|^2)^2}\Big]. \tag{7.5}$$

This theorem gives an estimate of the eventual energy “cascade” from motions of big length to motions of small length. This estimate of the energy transfer depends on δ, in such a way that it approaches 0 as δ approaches 0.

Proof:

As $\int_{\mathbb{T}^2}(v\cdot\nabla)v\cdot v dx = 0$, that is

$$\frac{1}{2}\sum_{k\in\mathbb{Z}^2_*}\sum_{j=1}^{2}\mathbb{E}\,\langle e_{j,k},(v\cdot\nabla)v\rangle\,\alpha_{j,k} = 0, \tag{7.6}$$

the N_δ expression becomes

$$N_\delta = \frac{1}{2}\sum_{k\in\mathbb{Z}^2_*}\sum_{j=1}^{2}\mathbb{E}\big[\frac{-2\delta\,|k|^2-\delta^2\,|k|^4}{(1+\delta\,|k|^2)^2}\,\langle e_{j,k},(v\cdot\nabla)v\rangle\,\alpha_{j,k}\big]. \tag{7.7}$$

On the other hand, from

$$(v\cdot\nabla)v = \frac{1}{4}\sum_{k'\in\mathbb{Z}^2_*}\sum_{k''\mathbb{Z}^2_*}\frac{k''^{\perp}(k'^{\perp}\cdot k'')}{|k'|\,|k''|}\times$$

$$\begin{aligned}
&\times\Big[\frac{\alpha_{1,k'}\alpha_{1,k''}-\alpha_{2,k'}\alpha_{2,k''}}{2(\sqrt{2}\pi)^2}\sin((k'+k'')\cdot x)+\\
&+\frac{\alpha_{1,k'}\alpha_{1,k''}+\alpha_{2,k'}\alpha_{2,k''}}{2(\sqrt{2}\pi)^2}\sin((k'-k'')\cdot x)+\\
&+\frac{\alpha_{1,k'}\alpha_{2,k''}-\alpha_{1,k''}\alpha_{2,k'}}{2(\sqrt{2}\pi)^2}\cos((k'-k'')\cdot x)+\\
&-\frac{\alpha_{1,k'}\alpha_{2,k''}+\alpha_{1,k''}\alpha_{2,k'}}{2(\sqrt{2}\pi)^2}\cos((k'+k'')\cdot x)\Big].
\end{aligned}\tag{7.8}$$

we obtain

$$\sum_{j=1}^{2}\langle e_{j,k},(v\cdot\nabla)v\rangle\,\alpha_{j,k} = I_1^{[k]}+I_2^{[k]}+I_3^{[k]}+I_4^{[k]},\tag{7.9}$$

where

$$I_1^{[k]} = -\frac{1}{4}\sum_{k'\in\mathbb{Z}^2_*,k'\neq k}\frac{(k-k')^{\perp}(k'^{\perp}\cdot(k-k'))}{|k'|\,|k-k'|}\cdot\frac{k^{\perp}}{|k|}\frac{\alpha_{1,k'}\alpha_{1,k-k'}-\alpha_{2,k'}\alpha_{2,k-k'}}{2\sqrt{2}\pi}\alpha_{1,k},$$

$$I_2^{[k]} = -\frac{1}{4}\sum_{k'\in\mathbb{Z}^2_*,k'\neq k}\frac{(k-k')^{\perp}(k'^{\perp}\cdot(k-k'))}{|k'|\,|k-k'|}\cdot\frac{k^{\perp}}{|k|}\frac{\alpha_{1,k'}\alpha_{1,k'-k}+\alpha_{2,k'}\alpha_{2,k'-k}}{2\sqrt{2}\pi}\alpha_{1,k},$$

$$I_3^{[k]} = -\frac{1}{4}\sum_{k'\in\mathbb{Z}^2_*,k'\neq k}\frac{(k-k')^{\perp}(k'^{\perp}\cdot(k-k'))}{|k'|\,|k-k'|}\cdot\frac{k^{\perp}}{|k|}\frac{\alpha_{1,k'}\alpha_{2,k-k'}+\alpha_{2,k'}\alpha_{1,k-k'}}{2\sqrt{2}\pi}\alpha_{2,k},$$

$$I_4^{[k]} = \frac{1}{4}\sum_{k'\in\mathbb{Z}^2_*,k'\neq k}\frac{(k-k')^{\perp}(k'^{\perp}\cdot(k-k'))}{|k'|\,|k-k'|}\cdot\frac{k^{\perp}}{|k|}\frac{\alpha_{1,k'}\alpha_{2,k'-k}-\alpha_{2,k'}\alpha_{1,k'-k}}{2\sqrt{2}\pi}\alpha_{2,k}.$$

From (7.7), (7.8) and (7.9) it follows that

(7.10)

$$N_\delta = \frac{1}{16\sqrt{2}\pi}\sum_{k\in\mathbb{Z}^2_*}\frac{2\delta\,|k|^2+\delta^2\,|k|^4}{(1+\delta\,|k|^2)^2}\sum_{k'\in\mathbb{Z}^2_*,k'\neq k}\frac{((k-k')^{\perp}\cdot k^{\perp})(k'^{\perp}\cdot(k-k'))}{|k|\,|k'|\,|k-k'|}\mathbb{E}\Phi_{k,k'},$$

where

$$\Phi_{k,k'} = (\alpha_{1,k'}\alpha_{1,k-k'} - \alpha_{2,k'}\alpha_{2,k-k'} + \alpha_{1,k'}\alpha_{1,k'-k} + \alpha_{2,k'}\alpha_{2,k'-k})\alpha_{1,k} + \\ + (\alpha_{1,k'}\alpha_{2,k-k'} + \alpha_{2,k'}\alpha_{1,k-k'} - \alpha_{1,k'}\alpha_{2,k'-k} + \alpha_{2,k'}\alpha_{1,k'-k})\alpha_{2,k}.$$

We set

$$A_k^2 = \alpha_{1,k}^2 + \alpha_{2,k}^2, \qquad k \in \mathbb{Z}_*^2. \tag{7.11}$$

As we have

$$|\Phi_{k,k'}| \leq 4A_k A_{k'} A_{k-k'}, \tag{7.12}$$

from (7.10) it follows that

$$|N_\delta| \leq C_1 \mathbb{E}\Big[\sum_{k\in\mathbb{Z}_*^2} \frac{2\delta\,|k|^2 + \delta^2\,|k|^4}{(1+\delta\,|k|^2)^2} \sum_{k'\in\mathbb{Z}_*^2, k'\neq k} |k-k'| A_k A_{k'} A_{k-k'}\Big] \tag{7.13}$$

with $C_1 = \dfrac{1}{4\sqrt{2}\pi}$.

Furthermore, thanks to Cauchy-Schwarz's inequality, for all $\varepsilon \in]0,1[$, we have

$$\sum_{k\in\mathbb{Z}_*^2} \sum_{k'\in\mathbb{Z}_*^2, k'\neq k} \frac{2\delta\,|k|^2 + \delta^2\,|k|^4}{(1+\delta\,|k|^2)^2}\,|k-k'|\,A_k A_{k'} A_{k-k'} \leq Q_\varepsilon^{\frac{1}{2}} R_\varepsilon^{\frac{1}{2}} \tag{7.14}$$

where

$$Q_\varepsilon = \sum_{k\in\mathbb{Z}_*^2} \Big(\sum_{k'\in\mathbb{Z}_*^2, k'\neq k} |k-k'|\,A_{k'} A_{k-k'}\,|k|^{-\varepsilon}\Big)^2,$$

$$R_\varepsilon = \sum_{k\in\mathbb{Z}_*^2} \Big(\frac{-2\delta\,|k|^2 - \delta^2\,|k|^4}{(1+\delta\,|k|^2)^2}\Big)^2 |k|^{2\varepsilon}\,A_k^2,$$

so it follows

$$|N_\delta| \leq C_1 \mathbb{E}\big(Q_\varepsilon^{\frac{1}{2}} R_\varepsilon^{\frac{1}{2}}\big). \tag{7.15}$$

Thanks to relation

$$\frac{1}{|k-k'|}\frac{1}{|k|^\varepsilon} \leq \frac{1}{|k-k'|^{1+\varepsilon}} + \frac{1}{|k|^{1+\varepsilon}}, \tag{7.16}$$

we have

$$Q_\varepsilon \le 2(Q_\varepsilon^{[1]} + Q_\varepsilon^{[2]}), \tag{7.17}$$

where

$$Q_\varepsilon^{[1]} = \sum_{k\in\mathbb{Z}^2_*} \Big(\sum_{k'\in\mathbb{Z}^2_*, k'\neq k} |k-k'|^2 A_{k-k'} A_{k'} \frac{1}{|k-k'|^{1+\varepsilon}} \Big)^2$$

$$Q_\varepsilon^{[2]} = \sum_{k\in\mathbb{Z}^2_*} \Big(\sum_{k'\in\mathbb{Z}^2_*, k'\neq k} |k-k'|^2 A_{k-k'} A_{k'} \frac{1}{|k|^{1+\varepsilon}} \Big)^2 .$$

As $\dfrac{1}{|k|^{1+\varepsilon}}$ doesn't depend on k', we have

$$\begin{aligned} Q_\varepsilon^{[2]} &= \sum_{k\in\mathbb{Z}^2_*} \frac{1}{|k|^{2(1+\varepsilon)}} \Big(\sum_{k'\in\mathbb{Z}^2_*, k'\neq k} |k-k'|^2 A_{k-k'} A_{k'} \Big)^2 \\ &\le \sum_{k\in\mathbb{Z}^2_*} \frac{1}{|k|^{2(1+\varepsilon)}} \Big(\sum_{k'\in\mathbb{Z}^2_*, k'\neq k} |k-k'|^4 A_{k-k'}^2 \Big) \Big(\sum_{k'\in\mathbb{Z}^2_*} A_{k'}^2 \Big). \end{aligned} \tag{7.18}$$

So, if we remember relations

$$\sum_{k'\in\mathbb{Z}^2_*, k'\neq k} |k-k'|^4 A_{k-k'}^2 = \|\Delta v\|_{L^2}^2 , \qquad \sum_{k'\in\mathbb{Z}^2_*} A_{k'}^2 = \|v\|_{L^2}^2 , \tag{7.19}$$

we get

$$Q_\varepsilon^{[2]} \le \sum_{k\in\mathbb{Z}^2_*} \frac{1}{|k|^{2(1+\varepsilon)}} \|v\|_{L^2}^2 \|\Delta v\|_{L^2}^2 . \tag{7.20}$$

Now we have to examine $Q_\varepsilon^{[1]}$. Thanks to Cauchy-Schwarz's inequality, we have

$$\begin{aligned} Q_\varepsilon^{[1]} \le \sum_{k\in\mathbb{Z}^2_*} \Big(\sum_{k'\in\mathbb{Z}^2_*, k'\neq k} A_{k'}^2 |k-k'|^2 A_{k-k'} \frac{1}{|k-k'|^{1+\varepsilon}} \Big) \times \\ \times \Big(\sum_{k'\in\mathbb{Z}^2_*, k'\neq k} |k-k'|^2 A_{k-k'} \frac{1}{|k-k'|^{1+\varepsilon}} \Big). \end{aligned}$$

We have

$$\sum_{k'\in\mathbb{Z}^2_*,k'\neq k} |k-k'|^2 A_{k-k'} \frac{1}{|k-k'|^{1+\varepsilon}} \leq$$
$$\leq \Big(\sum_{k'\in\mathbb{Z}^2_*,k'\neq k} |k-k'|^4 A^2_{k-k'} \Big)^{\frac{1}{2}} \Big(\sum_{k'\in\mathbb{Z}^2_*,k'\neq k} \frac{1}{|k-k'|^{2(1+\varepsilon)}} \Big)^{\frac{1}{2}} \leq$$
$$\leq \Big(\sum_{k\in\mathbb{Z}^2_*} \frac{1}{|k|^{2(1+\varepsilon)}} \Big)^{\frac{1}{2}} \|\Delta v\|_{L^2} .$$

On the other hand it holds

$$\sum_{k\in\mathbb{Z}^2_*} \Big(\sum_{k'\in\mathbb{Z}^2_*,k'\neq k} A^2_{k'} |k-k'|^2 A_{k-k'} \frac{1}{|k-k'|^{1+\varepsilon}} \Big) =$$
$$= \sum_{k'\in\mathbb{Z}^2_*} A^2_{k'} \Big(\sum_{k\in\mathbb{Z}^2_*,k\neq k'} |k-k'|^2 A_{k-k'} \frac{1}{|k-k'|^{1+\varepsilon}} \Big) \leq$$
$$\leq \sum_{k'\in\mathbb{Z}^2_*} A^2_{k'} \Big(\sum_{k\in\mathbb{Z}^2_*,k\neq k'} |k-k'|^4 A^2_{k-k'} \Big)^{\frac{1}{2}} \Big(\sum_{k\in\mathbb{Z}^2_*,k\neq k'} \frac{1}{|k-k'|^{2(1+\varepsilon)}} \Big)^{\frac{1}{2}} \leq \tag{7.21}$$
$$\leq \Big(\sum_{k\in\mathbb{Z}^2_*} \frac{1}{|k|^{2(1+\varepsilon)}} \Big)^{\frac{1}{2}} \|v\|^2_{L^2} \|\Delta v\|_{L^2} .$$

It follows that

$$Q^{[1]}_\varepsilon \leq \sum_{k\in\mathbb{Z}^2_*} \frac{1}{|k|^{2(1+\varepsilon)}} \|v\|^2_{L^2} \|\Delta v\|^2_{L^2} . \tag{7.22}$$

From (7.20) and (7.22) results

$$Q_\varepsilon \leq 2 \sum_{k\in\mathbb{Z}^2_*} \frac{1}{|k|^{2(1+\varepsilon)}} \|v\|^2_{L^2} \|\Delta v\|^2_{L^2} . \tag{7.23}$$

For R_ε finally we have

$$R^{\frac{1}{2}}_\varepsilon \leq \Big(\sum_{k\in\mathbb{Z}^2_*} \Big(\frac{2\delta |k|^2 + \delta^2 |k|^4}{(1+\delta |k|^2)^2} \Big)^2 |k|^{2\varepsilon} A^2_k \Big)^{\frac{1}{2}} \leq$$
$$\leq \Big(\sup_{k\in\mathbb{Z}^2_*} \frac{1}{|k|^{2(1-\varepsilon)}} \Big(\frac{2\delta |k|^2 + \delta^2 |k|^4}{(1+\delta |k|^2)^2} \Big)^2 \Big)^{\frac{1}{2}} \Big(\sum_{k\in\mathbb{Z}^2_*} |k|^2 A^2_k \Big)^{\frac{1}{2}} \leq \tag{7.24}$$
$$\leq \Big(\sup_{k\in\mathbb{Z}^2_*} \frac{1}{|k|^{2(1-\varepsilon)}} \Big(\frac{2\delta |k|^2 + \delta^2 |k|^4}{(1+\delta |k|^2)^2} \Big)^2 \Big)^{\frac{1}{2}} \|\nabla v\|_{L^2} .$$

We can prove that [7]

$$\mathbb{E}\big(\,\|v\|^2_{L^2(\mathbb{T}^2)}\,\|\nabla v\|^2_{L^2(\mathbb{T}^2)}\,\big) \le \frac{3}{2\nu^2}\Big(\sum_{(j,k)\in\mathbb{L}} \lambda^2_{j,k}\Big)^2. \tag{7.25}$$

and

$$\nu\mathbb{E}\,\|\Delta v\|^2_{L^2(\mathbb{T}^2)} = \frac{1}{2}\sum_{(j,k)\in\mathbb{L}} |k|^2\lambda^2_{j,k}. \tag{7.26}$$

From (7.25), (7.26), (7.15), (7.23), (7.24) it follows

$$\begin{aligned} |N_\delta| &\le C_1 f(\delta)\mathbb{E}[\|v\|_{L^2}\,\|\nabla v\|_{L^2}\,\|\Delta v\|_{L^2}] \\ &\le C_2 f(\delta)\sum_{(j,k)\in\mathbb{L}} \lambda^2_{j,k}\Big(\sum_{(j,k)\in\mathbb{L}} |k|^2\lambda^2_{j,k}\Big)^{\frac{1}{2}}. \end{aligned} \tag{7.27}$$

□

8 Three dimensional case

The stochastic approach to the $3D$ turbulence problem presents some technical difficulties, so we study the deterministic Navier-Stokes equations system.

Deterministic equation

We consider the $3D$ Navier-Stokes equations system

$$\partial_t v + (v \cdot \nabla) v + \nabla p - \nu \Delta v = f, \tag{8.1}$$

$$\nabla \cdot v = 0$$

$t \in \mathbb{R}^+$, $x \in \mathbb{R}^3$.

Like in $2D$ case, we choose as domain the $3D$ torus $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$. To study the (8.1), we present again the same functional setting choosen in the $2D$ case. So we set

$$\tilde{V}^\infty = \{v \in C^\infty(\mathbb{T}^3; \mathbb{R}^3) \,|\, \nabla \cdot v = 0, \int_{\mathbb{T}^3} v dx = 0\} \tag{8.2}$$

and we define the Hilbert space

$$\tilde{\mathcal{H}} = \overline{\tilde{V}^\infty}^{L^2(\mathbb{T}^3;\mathbb{R}^3)}. \tag{8.3}$$

We introduce the orthogonal projection operator $\mathcal{P}_{\tilde{\mathcal{H}}}$ from $L^2(\mathbb{T}^3; \mathbb{R}^3)$ on $\tilde{\mathcal{H}}$. Also in $3D$ case the following properties hold

$$\mathcal{P}_{\tilde{\mathcal{H}}} \nabla p = 0, \tag{8.4}$$

$$\mathcal{P}_{\tilde{\mathcal{H}}} \Delta v = \Delta v.$$

The system (8.1) can be written in the form

$$\frac{d}{dt} v + \mathcal{P}_{\tilde{\mathcal{H}}} (v \cdot \nabla) v - \nu \Delta v = \mathcal{P}_{\tilde{\mathcal{H}}} f. \tag{8.5}$$

The associated stationary system

$$\mathcal{P}_{\tilde{\mathcal{H}}} (v \cdot \nabla) v - \nu \Delta v = \mathcal{P}_{\tilde{\mathcal{H}}} f \tag{8.6}$$

admits at least a solution, called *stationary solution* of (8.5) [6].

Fourier series expression

Any periodic function $v \in L^2(\mathbb{T}^3;\mathbb{R}^3)$ can be developed using Fourier series.

Setting

(8.7)

$$\mathbb{Z}^3_1 = \{\, k = (k_1,k_2,k_3) \in \mathbb{Z}^3 \mid k_1 > 0 \vee (k_1 = 0, k_2 > 0) \vee (k_1 = 0, k_2 = 0, k_3 > 0) \,\},$$

we have

$$v = \sum_{k\in\mathbb{Z}^3_1} (\alpha_{1,k}\frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}} + \alpha_{2,k}\frac{\cos(k\cdot x)}{2\pi\sqrt{\pi}}), \tag{8.8}$$

where $\alpha_{j,k}$ belongs to $\mathbb{R}^3$, $j = 1,2$, $k \in \mathbb{Z}^3_1$.

From (8.8) follows

$$\|v\|^2_{L^2(\mathbb{T}^3)} = \sum_{k\in\mathbb{Z}^3_1} (|\alpha_{1,k}|^2 + |\alpha_{2,k}|^2) = \sum_{(j,k)\in\mathbb{L}} |\alpha_{j,k}|^2,$$

where from now on we define $\mathbb{L} = \{1,2\} \times \mathbb{Z}^3_1$.

Such a function v belongs to $\tilde{\mathcal{H}}$ if and only if

$$\alpha_{1,k}\cdot k = 0, \qquad \alpha_{2,k}\cdot k = 0 \tag{8.9}$$

(it follows from the divergence free request).

Observation: Conditions (8.9) mean that $\forall k \neq (0,0,0)$ vectors $\alpha_{1,k}$ and $\alpha_{2,k}$ belong to the bi-dimensional subspace orthogonal to the vector k. So, it is possible to fix a basis of orthogonal versors $e_{1,k}$, $e_{2,k}$ on a such subspace and to express $\alpha_{1,k}$ and $\alpha_{2,k}$ as a linear combination using this basis. In this way we have the following expression of v

(8.10)

$$\begin{aligned} v &= \sum_{k\in\mathbb{Z}^3_1} [(\alpha_{1,k,1}e_{1,k} + \alpha_{1,k,2}e_{2,k})\frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}} + (\alpha_{2,k,1}e_{1,k} + \alpha_{2,k,2}e_{2,k})\frac{\cos(k\cdot x)}{2\pi\sqrt{\pi}}] \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}^3_*} [(\alpha_{1,k,1}e_{1,k} + \alpha_{1,k,2}e_{2,k})\frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}} + (\alpha_{2,k,1}e_{1,k} + \alpha_{2,k,2}e_{2,k})\frac{\cos(k\cdot x)}{2\pi\sqrt{\pi}}] \end{aligned}$$

where $\mathbb{Z}^3_* = \mathbb{Z}^3 \backslash \{(0,0,0)\}$.

Then

$$(e_{j,k}\frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}}, e_{j,k}\frac{\cos(k\cdot x)}{2\pi\sqrt{\pi}}), \tag{8.11}$$

with $j=1,2$ and $k\in\mathbb{Z}_1^3$ is an orthonormal basis of $\tilde{\mathcal{H}}$ in $3D$ case.

From (8.10) we obtain the Fourier series expression of **the nonlinear term**

$$\begin{aligned}
&\mathcal{P}_{\tilde{\mathcal{H}}}(v\cdot\nabla)v = \\
&= \frac{1}{32\pi^3}\sum_{k'\in Z_*^3}\sum_{k''\in Z_*^3}[(k''\cdot\alpha_{1,k'})\alpha_{1,k''}-(k''\cdot\alpha_{2,k'})\alpha_{2,k''}]\sin((k'+k'')\cdot x)+\\
&+[(k''\cdot\alpha_{1,k'})\alpha_{1,k''}+(k''\cdot\alpha_{2,k'})\alpha_{2,k''}]\sin((k'-k'')\cdot x)+\\
&+[(k''\cdot\alpha_{2,k'})\alpha_{1,k''}+(k''\cdot\alpha_{1,k'})\alpha_{2,k''}]\cos((k'+k'')\cdot x)+\\
&+[(k''\cdot\alpha_{2,k'})\alpha_{1,k''}-(k''\cdot\alpha_{1,k'})\alpha_{2,k''}]\cos((k'-k'')\cdot x).
\end{aligned} \tag{8.12}$$

We can prove that the following equality holds again on the $3D$ torus

$$\langle(v\cdot\nabla)v, v\rangle = 0, \tag{8.13}$$

at the contrary the equality

$$\langle(v\cdot\nabla)v, \Delta v\rangle = 0 \tag{8.14}$$

doesn't hold.

9 Kinetic energy balance

Scalarly multiplying by v the two members of the equation (8.5) and integrating from 0 to t, thanks to (8.13) we obtain

$$\frac{1}{2}\|v(t)\|^2_{L^2(\mathbb{T}^3)} - \frac{1}{2}\|v(0)\|^2_{L^2(\mathbb{T}^3)} + \nu\int_0^t\|\nabla v\|^2_{L^2(\mathbb{T}^3)}dt' = \int_0^t\langle f, v\rangle dt'. \tag{9.1}$$

In particular, if v is the stationary solution, then we have

$$\nu\|\nabla v\|^2_{L^2(\mathbb{T}^3)} = \langle f, v\rangle, \tag{9.2}$$

which Fourier series expression is

$$\nu \sum_{(j,k)\in\mathbb{L}} |k|^2 \,|\alpha_{j,k}|^2 = \sum_{(j,k)\in\mathbb{L}} f_{j,k}\cdot\alpha_{j,k}, \tag{9.3}$$

where we set $f_{1,k} = \Big\langle f, \dfrac{\sin(k\cdot x)}{2\pi\sqrt{\pi}}\Big\rangle$ e $f_{2,k} = \Big\langle f, \dfrac{\cos(k\cdot x)}{2\pi\sqrt{\pi}}\Big\rangle$.

We now scalarly multiply by Δv the two members of the equation (8.5) and we integrate from 0 to t, we obtain

$$\begin{aligned} &\frac{1}{2}\|\nabla v(t)\|^2_{L^2(\mathbb{T}^3)} - \frac{1}{2}\,\|\nabla v(0)\|^2_{L^2(\mathbb{T}^3)} - \int_0^t \big\langle (v\cdot\nabla)v, \Delta v\big\rangle dt' \\ &+ \nu \int_0^t \|\Delta v\|^2_{L^2(\mathbb{T}^3)} dt' = -\int_0^t \big\langle f, \Delta v\big\rangle dt'. \end{aligned} \tag{9.4}$$

In particular, if v is the stationary solution, then we have

$$\nu\,\|\Delta v\|^2_{L^2(\mathbb{T}^3)} - \big\langle (v\cdot\nabla)v, \Delta v\big\rangle = \big\langle f, -\Delta v\big\rangle. \tag{9.5}$$

As already observed, in $3D$ the property

$$\langle (v\cdot\nabla)v, \Delta v\rangle = 0 \tag{9.6}$$

doesn't hold.
In the deterministic case, unlike the stochastic one, it is easily possible to find an estimate for this term. In fact we can prove

$$|\langle (v\cdot\nabla)v, \Delta v\rangle| \le \frac{\nu}{2}\,\|\Delta v\|^2_{L^2(\mathbb{T}^3)} + \frac{2}{\nu^3} c\,\|\nabla v\|^6_{L^2(\mathbb{T}^3)}. \tag{9.7}$$

Idea of proof:
Thanks to Hölder's inequality, we can obtain

$$|\langle (v\cdot\nabla)v, \Delta v\rangle| = \|\nabla v\|_{L^2(\mathbb{T}^3)}\,\|\nabla v\|^2_{L^4(\mathbb{T}^3)}\,. \tag{9.8}$$

Thanks to Hölder and Sobolev's inequality it is possible to give an upper estimate of $\|\nabla v\|^2_{L^4(\mathbb{T}^3)}$ in this way

$$\|\nabla v\|^2_{L^4(\mathbb{T}^3)} \leq c\,\|\nabla v\|^{\frac{1}{2}}_{L^2(\mathbb{T}^3)}\,\|\Delta v\|^{\frac{3}{2}}_{L^2(\mathbb{T}^3)} \tag{9.9}$$

and so to obtain

$$|\langle(v\cdot\nabla)v,\Delta v\rangle| \leq c\,\|\nabla v\|^{\frac{3}{2}}_{L^2(\mathbb{T}^3)}\,\|\Delta v\|^{\frac{3}{2}}_{L^2(\mathbb{T}^3)}. \tag{9.10}$$

Finally, using Young's inequality we obtain

$$c\,\|\nabla v\|^{\frac{3}{2}}_{L^2(\mathbb{T}^3)}\,\|\Delta v\|^{\frac{3}{2}}_{L^2(\mathbb{T}^3)} \leq \frac{\nu}{2}\,\|\Delta v\|^2_{L^2(\mathbb{T}^3)} + \frac{2}{\nu^3}c\,\|\nabla v\|^6_{L^2(\mathbb{T}^3)}\,. \tag{9.11}$$

□

From

$$\nu\,\|\Delta v\|^2_{L^2(\mathbb{T}^3)} - \big\langle(v\cdot\nabla)v,\Delta v\big\rangle = \big\langle f,-\Delta v\big\rangle \tag{9.12}$$

and

$$|\langle(v\cdot\nabla)v,\Delta v\rangle| \leq \frac{\nu}{2}\,\|\Delta v\|^2_{L^2(\mathbb{T}^3)} + \frac{2}{\nu^3}c\,\|\nabla v\|^6_{L^2(\mathbb{T}^3)}. \tag{9.13}$$

it follows an estimate for $\nu\,\|\Delta v\|^2_{L^2(\mathbb{T}^3)}$

$$\frac{\nu}{2}\,\|\Delta v\|^2_{L^2(\mathbb{T}^3)} \leq \frac{2}{\nu^3}c\,\|\nabla v\|^6_{L^2(\mathbb{T}^3)} + \|\nabla f\|_{L^2(\mathbb{T}^3)}\,\|\nabla v\|_{L^2(\mathbb{T}^3)}\,, \tag{9.14}$$

its Fourier series expression is

$$\frac{\nu}{2}\sum_{j,k\in\mathbb{L}}|k|^4\,|\alpha_{j,k}|^2 \leq \frac{2}{\nu^3}c\Big(\sum_{j,k\in\mathbb{L}}|k|^2\,|\alpha_{j,k}|^2\Big)^3 + \Big(\sum_{j,k\in\mathbb{L}}|k|^2\,|f_{j,k}|^2\Big)^{\frac{1}{2}}\Big(\sum_{j,k\in\mathbb{L}}|k|^2\,|\alpha_{j,k}|^2\Big)^{\frac{1}{2}}. \tag{9.15}$$

10 Local average

Let Θ_δ be the family of functions

$$\Theta_\delta(x) = \mathcal{F}^{-1}\Big(\frac{1}{1+\delta|\xi|^2}\Big)(x), \qquad \delta > 0. \tag{10.1}$$

We consider the convolution operator $\Theta_\delta *$ already used to define the local average in $2D$ case.

We can observe that the function $\Theta_\delta(x)$ has the $3D$ explicit form

$$\Theta_\delta(x) = \frac{1}{4\pi\delta\,|x|} e^{-\frac{|x|}{\sqrt{\delta}}}. \tag{10.2}$$

Also in $3D$ case we can observe that

$$\|\Theta_\delta * \varphi\|_{L^\infty(\mathbb{R}^3)} \leq \|\Theta_\delta\|_{L^1(\mathbb{R}^3)} \|\varphi\|_{L^\infty(\mathbb{R}^3)} = \|\varphi\|_{L^\infty(\mathbb{R}^3)}, \tag{10.3}$$

then it is possible to define the local average operator $\Theta_\delta *$ over the class of functions $L^\infty(\mathbb{R}^3)$.

The basis elements $e_{j,k}\dfrac{\sin(k\cdot x)}{2\pi\sqrt{\pi}}$, $e_{j,k}\dfrac{\cos(k\cdot x)}{2\pi\sqrt{\pi}}$, with $j = 1, 2$, belong to $L^\infty(\mathbb{R}^3)$, so we have

$$\Theta_\delta * e_{j,k}\frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}} = \frac{1}{1+\delta\,|k|^2} e_{j,k}\frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}}, \qquad j = 1, 2, \quad k \in \mathbb{Z}^3_1, \tag{10.4}$$

similary to $e_{j,k}\dfrac{\cos(k\cdot x)}{2\pi\sqrt{\pi}}$.

Using the relation (10.4) we define the operator $\Theta_\delta *$ on $\tilde{\mathcal{H}}$.

Indeed, if $v \in \tilde{\mathcal{H}}$,

(10.5)
$$\begin{aligned}\Theta_\delta * v &= \sum_{k\in\mathbb{Z}^3_1} \Big(\big\langle v, \frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}} \big\rangle \Theta_\delta * \frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}} + \big\langle v, \frac{\cos(k\cdot x)}{2\pi\sqrt{\pi}} \big\rangle \Theta_\delta * \frac{\cos(k\cdot x)}{2\pi\sqrt{\pi}} \Big) \\ &= \sum_{k\in\mathbb{Z}^3_1} \Big(\frac{\big\langle v, \frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}} \big\rangle}{1+\delta\,|k|^2} \frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}} + \frac{\big\langle v, \frac{\cos(k\cdot x)}{2\pi\sqrt{\pi}} \big\rangle}{1+\delta\,|k|^2} \frac{\cos(k\cdot x)}{2\pi\sqrt{\pi}} \Big).\end{aligned}$$

So we also set in 3-dimensional case $\overline{v} = \Theta_\delta * v$ and $u = v - \overline{v}$.

11 The macrocomponent energy balance

If we apply the local average operator to the Navier-Stokes equation (8.5), than we scalarly moltiply by $\overline{v}$ and integrate from 0 to t, we obtain the following macrocomponent energy equation:

$$
\begin{aligned}
&\frac{1}{2}\left\|\overline{v}(t)\right\|^2_{L^2(\mathbb{T}^3)} - \frac{1}{2}\left\|\overline{v}(0)\right\|^2_{L^2(\mathbb{T}^3)} + \int_0^t \langle \Theta_\delta * \mathcal{P}_{\tilde{\mathcal{H}}}\left(v\cdot\nabla\right)v, \overline{v}\rangle dt' \\
&+ \nu \int_0^t \left\|\nabla\overline{v}\right\|^2_{L^2(\mathbb{T}^3)} dt' = \int_0^t \langle \overline{f}, \overline{v}\rangle dt'.
\end{aligned} \tag{11.1}
$$

If $\overline{v}$ is the stationary solution, we have

$$
\langle \Theta_\delta * \mathcal{P}_{\tilde{\mathcal{H}}}\left(v\cdot\nabla\right)v, \overline{v}\rangle + \nu\left\|\nabla\overline{v}\right\|^2_{L^2(\mathbb{T}^3)} = \left\langle \overline{f}, \overline{v}\right\rangle, \tag{11.2}
$$

which Fourier series expression is

(11.3)

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^3_1} \frac{1}{(1+\delta\left|k\right|^2)^2}\left[\alpha_{1,k}\cdot\left\langle (v\cdot\nabla)v, \frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}}\right\rangle + \alpha_{2,k}\cdot\left\langle (v\cdot\nabla)v, \frac{\cos(k\cdot x)}{2\pi\sqrt{\pi}}\right\rangle\right] + \\
&+\nu\sum_{(j,k)\in\mathbb{L}} \frac{\left|k\right|^2}{(1+\delta\left|k\right|^2)^2}\left|\alpha_{j,k}\right|^2 = \sum_{(j,k)\in\mathbb{L}} \frac{1}{(1+\delta\left|k\right|^2)^2} f_{j,k}\cdot\alpha_{j,k}.
\end{aligned}
$$

So the equality (11.2), or (11.3) as well, gives the possibility to interpret the phenomenon of the energy "cascade" from local average motion $\overline{v} = \Theta_\delta * v$ to fluctuations $u = v - \overline{v}$ by means of the nonlinear term.

12 An estimate of the nonlinear term in the $3D$ case

Theorem: *We suppose that v is a stationary solution of the Navier-Stokes equations*

$$\frac{d}{dt}v + \mathcal{P}_{\tilde{\mathcal{H}}}(v\cdot\nabla)v - \nu\Delta v = \mathcal{P}_{\tilde{\mathcal{H}}}f. \tag{12.1}$$

Then there exists a function $g(\delta)$, $\delta > 0$, such that

$$g(\delta) \to 0 \qquad \textit{as} \quad \delta \to 0$$

and that

$$M_\delta + \nu \sum_{(j,k)\in\mathbb{L}} \frac{|k|^2}{(1+\delta|k|^2)^2}|\alpha_{j,k}|^2 = \sum_{(j,k)\in\mathbb{L}} \frac{1}{(1+\delta|k|^2)^2} f_{j,k}\cdot\alpha_{j,k}. \tag{12.2}$$

(12.3)

$$|M_\delta| \le C_1 g(\delta)\|v\|_{L^2(\mathbb{T}^3)}\|\Delta v\|^2_{L^2(\mathbb{T}^3)} \le C_2 g(\delta)\big(\sum_{(j,k)\in\mathbb{L}}|\alpha_{j,k}|^2\big)^{\frac12}\times$$
$$\times\Big[\frac{2}{\nu^3}c\big(\sum_{(j,k)\in\mathbb{L}}|k|^2|\alpha_{j,k}|^2\big)^3 + \big(\sum_{(j,k)\in\mathbb{L}}|k|^2|f_{j,k}|^2\big)^{\frac12}\big(\sum_{(j,k)\in\mathbb{L}}|k|^2|\alpha_{j,k}|^2\big)^{\frac12}\Big],$$

where

(12.4)

$$M_\delta = \frac12\sum_{k\in\mathbb{Z}^3_*}\frac{1}{(1+\delta|k|^2)^2}\Big[\Big\langle (v\cdot\nabla)v, \alpha_{1,k}\frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}}\Big\rangle + \Big\langle (v\cdot\nabla)v, \alpha_{2,k}\frac{\cos(k\cdot x)}{2\pi\sqrt{\pi}}\Big\rangle\Big]$$

and with constants C_1, C_2 independent from δ. More precisely, the function $g(\delta)$ is given by

$$g(\delta) = \inf_{0<\varepsilon<1}\Big[\Big(\sum_{k'\in\mathbb{Z}^3_*}\frac{1}{|k'|^{3+\varepsilon}}\Big)^{\frac12}\Big(\sup_{k\in\mathbb{Z}^3_*}\frac{1}{|k|^{\frac{7-\varepsilon}{2}}}\Big(\frac{2\delta|k|^2+\delta^2|k|^4}{(1+\delta|k|^2)^2}\Big)^2\Big)^{\frac12}\Big], \tag{12.5}$$

where $\mathbb{Z}^3_* = \mathbb{Z}^3\backslash\{(0,0,0)\}$.

Proof:

Similary to what done in $2D$ case, from (8.12) and from (8.11) we obtain

(12.6)
$$\left\langle (v\cdot\nabla)v, \alpha_{1,k}\frac{\sin(k\cdot x)}{2\pi\sqrt{\pi}}\right\rangle + \left\langle (v\cdot\nabla)v, \alpha_{2,k}\frac{\cos(k\cdot x)}{2\pi\sqrt{\pi}}\right\rangle = I_1^{[k]}+I_2^{[k]}+I_3^{[k]}+I_4^{[k]}$$

where

$$I_1^{[k]} = \frac{1}{4}\sum_{k'\in Z_*^3, k'\neq k}\frac{(k-k')\cdot\alpha_{1,k'}\alpha_{1,k-k'}-(k-k')\cdot\alpha_{2,k'}\alpha_{2,k-k'}}{2(2\pi\sqrt{\pi})}\cdot\alpha_{1,k},$$

$$I_2^{[k]} = \frac{1}{4}\sum_{k'\in Z_*^3, k'\neq k}\frac{(k'-k)\cdot\alpha_{1,k'}\alpha_{1,k'-k}+(k'-k)\cdot\alpha_{2,k'}\alpha_{2,k'-k}}{2(2\pi\sqrt{\pi})}\cdot\alpha_{1,k},$$

$$I_3^{[k]} = \frac{1}{4}\sum_{k'\in Z_*^3, k'\neq k}\frac{(k-k')\cdot\alpha_{2,k'}\alpha_{1,k-k'}+(k-k')\cdot\alpha_{1,k'}\alpha_{2,k-k'}}{2(2\pi\sqrt{\pi})}\cdot\alpha_{2,k},$$

$$I_4^{[k]} = \frac{1}{4}\sum_{k'\in Z_*^3, k'\neq k}\frac{(k'-k)\cdot\alpha_{2,k'}\alpha_{1,k'-k}-(k'-k)\cdot\alpha_{1,k'}\alpha_{2,k'-k}}{2(2\pi\sqrt{\pi})}\cdot\alpha_{2,k}.$$

From (12.4) and from (12.6) it follows that

(12.7)
$$M_\delta = \frac{1}{32\pi\sqrt{\pi}}\sum_{k\in Z_*^3}\frac{-2\delta|k|^2-\delta^2|k|^4}{(1+\delta|k|^2)^2}\sum_{k'\in Z_*^3, k'\neq k}\Phi_{k,k'}$$

where

(12.8)
$$\Phi_{k,k'} = [(k-k')\cdot\alpha_{1,k'}\alpha_{1,k-k'}-(k-k')\cdot\alpha_{2,k'}\alpha_{2,k-k'}+(k'-k)\cdot\alpha_{1,k'}\alpha_{1,k'-k}+ \\ +(k'-k)\cdot\alpha_{2,k'}\alpha_{2,k'-k}]\cdot\alpha_{1,k}+[(k-k')\cdot\alpha_{2,k'}\alpha_{1,k-k'}+(k-k')\cdot\alpha_{1,k'}\alpha_{2,k-k'}+ \\ +(k'-k)\cdot\alpha_{2,k'}\alpha_{1,k'-k}-(k'-k)\cdot\alpha_{1,k'}\alpha_{2,k'-k}]\cdot\alpha_{2,k}.$$

Setting

(12.9)
$$A_k^2 = |\alpha_{1,k}|^2+|\alpha_{2,k}|^2,$$

we have

(12.10)
$$|\Phi_{k,k'}| \leq 4|k-k'|A_kA_{k'}A_{k-k'},$$

so it follows that

(12.11)
$$|M_\delta| \leq C_1\Big[\sum_{k\in\mathbb{Z}_*^3}\frac{2\delta|k|^2+\delta^2|k|^4}{(1+\delta|k|^2)^2}\sum_{k'\in\mathbb{Z}_*^3, k'\neq k}|k-k'|A_kA_{k'}A_{k-k'}\Big]$$

with $C_1 = \dfrac{1}{8\pi\sqrt{\pi}}$.

Making use of analogous calculations like in $2D$ case, we obtain the result

$$|M_\delta| \leq C_1 g(\delta) \left\| v \right\|_{L^2(\mathbb{T}^3)} \left\| \Delta v \right\|^2_{L^2(\mathbb{T}^3)} . \tag{12.12}$$

□

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