A trajectorial interpretation of entropy dissipation and a non intrinsic Bakry Emery criterion for SDEs

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Motivations:

- Probabilistic interpretation of the decrease of entropy for a Markov process?

  What is the limit?

- Trajectorial meaning of the dissipation of entropy for diffusion processes?

- By working at the level of trajectories of the diffusion, can we tell something (new) about dissipation of entropy dissipation ?

Outline

1. Entropy decrease for Markov processes : a backward point of view
2. Entropy dissipation for diffusion processes : a pathwise description
3. Dissipation of entropy dissipation: a non intrinsic Bakry-Emery criterion

Entropy decrease for Markov processes

H0) $U : [0,\infty) \to \mathbb{R}$ is a convex function such that $\inf U > -\infty$.

We consider

- $(X_t : t \geq 0)$ continuous-time Markov process with values in $(E, \mathcal{E})$.
- P_0, Q_0 probability measures on E.
- $(X_t^{P_0}, t \geq 0)$ and $(X_t^{Q_0}, t \geq 0)$ realizations of (X_t) with $X_0^{P_0} \sim P_0$ and $X_0^{Q_0} \sim Q_0$ respectively.
- P_t and Q_t, $t \geq 0$ laws of $X_t^{P_0}$ and $X_t^{Q_0}$ respectively.
- $H_U(p|q) = \begin{cases} \int_E U\left(\frac{dp}{dq}(x)\right) dq(x) & \text{if } p \ll q \\ +\infty \text{ otherwise.} \end{cases}$ $U-$ relative entropy.

Examples: $U(r) = r\ln(r)$, $U(r) = (r-1)^2$, $U(r) = |r-1|$.

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Proposition

If for some $t \geq 0$, $P_t \ll Q_t$, then :

- $\mathcal{L}(X_r^{P_0} : r \geq t) \ll \mathcal{L}(X_r^{Q_0} : r \geq t)$ with density $\frac{dP_t}{dQ_t}(X_t^{Q_0})$
- for all $s \geq t$, $P_s \ll Q_s$,
- $\left(\frac{dP_s}{dQs}(X_s^{Q_0})\right)_{s \geq t}$ is a backward martingale with respect to the filtration $\mathcal{F}_s = \sigma(X_r^{Q_0}, r \geq s)$.

If moreover $H_U(P_t|Q_t) < +\infty$ for some $t \geq 0$, then $\left(U(\frac{dP_s}{dQ_s}(X_s^{Q_0}))\right)_{s \geq t}$ is a backward submartingale with respecto to $\mathcal{F}_s$.

In particular $t \in \mathbb{R}_+ \mapsto H_U(P_t|Q_t) \in \mathbb{R} \cup \{+\infty\}$ is non-increasing.

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Key point: if $P_t \ll Q_t$,

$$\mathbb{E}(f(X_r^{P_0}, r \geq t)) = \int_{\mathbb{R}^d} \mathbb{E}^{t,x}(f(X_r, r \geq t))P_t(dx) =$$

$$\int_{\mathbb{R}^d} \mathbb{E}^{t,x}\left(f(X_r, r \geq t)\frac{dP_t}{dQ_t}(X_0)\right) Q_t(dx) = \mathbb{E}(f(X_r^{Q_0}, r \geq t)\frac{dP_t}{dQ_t}(X_t^{Q_0})).$$

For $s \geq t$,

$$\mathbb{E}\left(f(X_r^{P_0}, r \geq s)\right) = \mathbb{E}\left(f(X_r^{Q_0}, r \geq s)\frac{dP_t}{dQ_t}(X_t^{Q_0})\right)$$

and also

$$\mathbb{E}\left(f(X_r^{P_0}, r \geq s)\right) = \mathbb{E}\left(f(X_r^{Q_0}, r \geq s)\frac{dP_s}{dQ_s}(X_s^{Q_0})\right).$$

Corollary

If $H_U(P_t|Q_t) < +\infty$ for some $t \geq 0$, then

$$\lim_{s\to\infty} H_U(P_s|Q_s) = \mathbb{E}\left(U\left(\lim_{s\to\infty} \frac{dP_s}{dQ_s}(X_s^{Q_0}) \right) \right) < \infty.$$

In particular, if $U(1) = 0$ and the tail σ-field $\cap_{s\geq 0}\mathcal{F}_s$ is trivial a.s. then $\lim_{s\to\infty} H_U(P_s|Q_s) = 0$.

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Diffusion process:

$$dX_t = b(t, X_t) + \sigma(t, X_t) dW_t \quad \in [0, T] \times \mathbb{R}^d$$

$b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$,
$\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \otimes d'}$
$a = \sigma\sigma^* : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \otimes d}$.

Goal: describe $\left(U(\frac{dP_s}{dQ_s}(X_s^{Q_0})) \right)_{0 \leq s \leq T}$.

Notation:

- $\mathbb{P}^T$, $\mathbb{Q}^T$, $\mathbb{P}^{T \to 0}$ and $\mathbb{Q}^{T \to 0}$: laws of

$$(X_t^{P_0}, t \leq T), (X_t^{Q_0}, t \leq T), (X_{T-t}^{P_0}, t \leq T) \text{ and } (X_{T-t}^{Q_0}, t \leq T)$$

- $(Y_t)_{t \leq T}$ canonical process on $C([0, T], \mathbb{R}^d)$ and $\mathcal{G}_t = \sigma(Y_s, 0 \leq s \leq t)$ its filtration.

In the canonical space:

- If $P_0 \ll Q_0$, then $\mathbb{P}^{T\to 0} \ll \mathbb{Q}^{T\to 0}$ with $\frac{d\mathbb{P}^{T\to 0}}{d\mathbb{Q}^{T\to 0}} = \frac{dP_0}{dQ_0}(Y_T)$.

$$D_t^T \stackrel{\text{def}}{=} \frac{d\mathbb{P}^{T\to 0}}{d\mathbb{Q}^{T\to 0}} |_{\mathcal{G}_t} = \frac{dP_{T-t}}{dQ_{T-t}}(Y_t), \quad 0 \leq t \leq T,$$

  is a $\mathbb{Q}^{T\to 0} - \mathcal{G}_t$ is a (uniformly integrable) martingale with a right continuous version also denoted D_t^T (Girsanov density).

- $H_U(P_s|Q_s) < +\infty$ if and only if $U\left(\frac{dP_{T-t}}{dQ_{T-t}}(Y_t)\right)$, $0 \leq t \leq T-s$, is a uniformly integrable $\mathbb{Q}^{T\to 0} - \mathcal{G}_t$ submartingale.

- Pathwise entropy $\mathbb{H}_U(\mathbb{P}^T|\mathbb{Q}^T) = H_U(P_0|Q_0) = \mathbb{H}_U(\mathbb{P}^{T\to 0}|\mathbb{Q}^{T\to 0})$.

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Time reversal of the diffusion property

Theorem [Haussmann Pardoux 86] [Millet, Nualart, Sanz 89]

Assume

$H1)$ σ, b globally Lipschitz (can be relaxed)

$H2)_{Q_0}$ for $t > 0$, Q_t has a density $q_t(x)$ w.r.t. the Lebesgue meas. on $\mathbb{R}^d$

$H3)_{Q_0}$ The distributional divergence $\partial_j(a_{ij}(t,x)q_t(x))$ satisfies

$$\int_0^T \int_D |\partial_j(a_{ij}(t,x)q_t(x))|dxdt < \infty \text{ for any bounded set } D \subset \mathbb{R}^d$$

then $\mathbb{Q}^{T\to 0}$ is a solution to the martingale problem $(MP)_{Q_0}$

$M_t^f := f(Y_t) - f(Y_0) - \int_0^t \frac{1}{2}\bar{a}_{ij}(s,Y_s)\partial_{ij}f(Y_s) + \bar{b}^i_{Q_0}(s,Y_s)\partial_i f(Y_s)ds$
is a martingale (for nice test functions f), where

- $\bar{a}_{ij}(t,x) := a_{ij}(T-t,x), i,j = 1,\dots,d,$
- $\bar{b}^i_{Q_0}(t,x) = -b^i(T-t,x) + \frac{\partial_j(a_{ij}(T-t,x)q_{T-t}(x))}{q_{T-t}(x)}$ (with $\frac{*}{0} = 0$.)

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Lemma If $H1)$, $H2)_{Q_0}$, $H3)_{Q_0}$, $H3)_{P_0}$, and $P_0 \ll Q_0$ hold, then $\nabla \ln \frac{p_t}{q_t}$ and $\nabla \frac{p_t}{q_t}$ have a meaning (as equivalence classes if a singular).
Assume moreover $\mathbb{Q}^{T\to 0}$ is an extremal solution to $(MP)_{Q_0}$. Then

i) Setting $M_t^i := Y_t^i - Y_0^i - \int_0^t \bar{b}^i_{Q_0}(s, Y_s)ds$ and $R := \inf\{s \in [0,T] : D_s^T = 0\}$ $(\mathcal{G}_t)$-stopping time $\mathbb{Q}^{T\to 0}$ $a.s$, $\forall t \in [0,T]$ we have

$$\begin{aligned} D_t^T &= \frac{p_T}{q_T}(Y_0) + \int_0^t \nabla \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R} \cdot dM_s \\ &= \frac{p_T}{q_T}(Y_0) + \int_0^t \nabla \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{\{\frac{p_{T-s}}{q_{T-s}}(Y_s)>0\}} \cdot dM_s \end{aligned}$$

and

$$\langle D^T \rangle_t = \int_0^t \left(\nabla \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\right)^* \bar{a}(s, Y_s) \nabla \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R}\, ds (< \infty).$$

Lemma If $H1)$, $H2)_{Q_0}$, $H3)_{Q_0}$, $H3)_{P_0}$, and $P_0 \ll Q_0$ hold, then $\nabla \ln \frac{p_t}{q_t}$ and $\nabla \frac{p_t}{q_t}$ have a meaning (as equivalence classes if a singular). Assume moreover $\mathbb{Q}^{T\to 0}$ is an extremal solution to $(MP)_{Q_0}$. Then

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$$

ii) $\mathbb{Q}^{T\to 0}$−a.s. $\forall t \in [0, T]$,

$$D_t^T = \mathbf{1}_{\{t<\tau\}} \frac{dp_T}{dq_T}(Y_0) \times \exp\Big\{ \int_0^t \nabla \Big[\ln \frac{p_{T-s}}{q_{T-s}}\Big](Y_s) \cdot dM_s - \frac{1}{2}\int_0^t \Big(\nabla \Big[\ln \frac{p_{T-s}}{q_{T-s}}\Big](Y_s)\Big)^* \bar{a}(s, Y_s) \nabla \Big[\ln \frac{p_{T-s}}{q_{T-s}}\Big](Y_s) ds \Big\}.$$

where

$$\tau := \text{ínf} \Big\{ t \in [0, T] : \int_0^t \Big(\nabla \Big[\ln \frac{p_{T-s}}{q_{T-s}}\Big](Y_s)\Big)^* \bar{a}(s, Y_s) \nabla \Big[\ln \frac{p_{T-s}}{q_{T-s}}\Big](Y_s) ds = \infty \Big\},$$

and one has $R = \tau \wedge \tau^o$ where $\tau^o := 0 \cdot \mathbf{1}_{D_0^T = 0} + \infty \cdot \mathbf{1}_{D_0^T > 0}$

Stochastic $U-$entropy dissipation formula

- U'_- left-hand derivative of U on $(0,+\infty)$
- $U''(dy)$ the second order distribution derivative on $(0,+\infty)$

THEOREM 1

Assume $H_U(P_0|Q_0)<\infty$, $H1)$, $H2)_{Q_0}$, $H3)_{Q_0}$, $H3)_{P_0}$ and $\mathbb{Q}^{T\to 0}$ extremal solution of $(MP)_{Q_0}$. The $\mathbb{Q}^{T\to 0}-\mathcal{G}_t$ submartingale $(U(D_t^T))_{t\in[0,T]}$ has Doob-Meyer decomposition

$$U(D_t^T)=U(D_0^T)+\int_0^t U'_-(D_s^T)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R}\cdot dM_s$$
$$+\frac{1}{2}\int_{(0,+\infty)} L_t^r(D^T)U''(dr)-1_{\{0<R\leq t\}}\Delta U(0),$$

where $R:=\inf\{s\in[0,T]:D_s^T=0\}$,$\Delta U(0)=\lim_{x\to 0^+}U(x)-U(0)\leq 0$ and $L_t^r(D^T)$ is the local time at level $r\geq 0$ and time t of $(D_s^T)_{s\in[0,T]}$.

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Assume $H_U(P_0|Q_0) < \infty$, $H1)$, $H2)_{Q_0}$, $H3)_{Q_0}$, $H3)_{P_0}$ and $\mathbb{Q}^{T\to 0}$ extremal solution of $(MP)_{Q_0}$. The $\mathbb{Q}^{T\to 0} - \mathcal{G}_t$ submartingale $(U(D_t^T))_{t\in[0,T]}$ has Doob-Meyer decomposition

$$U(D_t^T) = U(D_0^T) + \int_0^t U'_-(D_s^T)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R}\cdot dM_s + \frac{1}{2}\int_{(0,+\infty)} L_t^r(D^T)U''(dr) - 1_{\{0<R\le t\}}\Delta U(0),$$

where $R := \text{ínf}\{s \in [0,T] : D_s^T = 0\}$,$\Delta U(0) = \text{lím}_{x\to 0^+} U(x) - U(0) \le 0$ and $L_t^r(D^T)$ is the local time at level $r \ge 0$ and time t of $(D_s^T)_{s\in[0,T]}$.

In particular, if U is continuous on $[0,+\infty)$ and C^2 on $(0,+\infty)$, $\forall t \in [0,T]$

$$U(D_t^T) = U(D_0^T) + \int_0^t U'(D_s^T)\nabla \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R} \cdot dM_s$$
$$+ \frac{1}{2}\int_0^t U''\left(\frac{p_{T-s}}{q_{T-s}}(Y_s)\right)\left(\nabla^* \left[\frac{p_{T-s}}{q_{T-s}}\right] \bar{a}(s,\cdot)\nabla \left[\frac{p_{T-s}}{q_{T-s}}\right]\right)(Y_s)\mathbf{1}_{s<R}ds.$$

U- entropy dissipation: $\forall t \in [0,T]$, denoting by $\tilde{\mathbb{E}}^{T\to 0}$ the expectation under $\mathbb{Q}^{T\to 0}$,

$$H_U(P_t|Q_t) = H_U(P_T|Q_T) - \Delta U(0)\mathbb{Q}^{T\to 0}(0 < R \leq T-t) + \frac{1}{2}\tilde{\mathbb{E}}^{T\to 0}\left(\int_{(0,+\infty)} L^r_{T-t}(D^T)U''(dr)\right).$$

Last, when U is continuous on $[0,+\infty)$ and C^2 on $(0,+\infty)$,

$$H_U(P_T|Q_T) = H_U(P_0|Q_0) - \frac{1}{2}\int_0^T \underbrace{\int_{\{\frac{p_s}{q_s}(x)>0\}} U''\left(\frac{p_s}{q_s}(x)\right)\left(\nabla^*\left[\frac{p_s}{q_s}\right]a(s,\cdot)\nabla\left[\frac{p_s}{q_s}\right]\right)(x)q_s(x)dx\,ds}_{I_U(p_s|q_s) \text{ U-Fischer information}}$$

Corollary: Dissipation of total variation

1) For the choice $U(x) = |x-1|$, under the assumptions of **Theorem 1** in particular that $P_0 \ll Q_0$

$$\forall t \leq T, \ \|P_t - Q_t\|_{\mathrm{TV}} = \|P_0 - Q_0\|_{\mathrm{TV}} - \tilde{\mathbb{E}}^{T\to 0}(L^1_T(D^T) - L^1_{T-t}(D^T)).$$

2) When $\frac{dp_t}{dq_t}(Y_t)$ is a continuous $\mathbb{Q}^T$-$\mathcal{G}_t$ semimartingale and in particular if $(t,x) \mapsto \frac{p_t}{q_t}(x)$ is well-defined and of class $C^{1,2}$, we deduce that

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where $\tilde{\mathbb{E}}^T$ denotes the expectation under $\mathbb{Q}^T$.

3) If $x \mapsto q_t(x)$ and $x \mapsto \frac{dP_t}{dQ_t}(x)$ are respectively of class C^1 and C^2 dt a.e., + integrability assumptions (valid e.g. if $\frac{dP_0}{dQ_0} \in L^2(Q_0)$) then, $\forall t \leq T$,

$$\|P_t - Q_t\|_{\mathrm{TV}} = \|P_0 - Q_0\|_{\mathrm{TV}} - \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \widetilde{sign}\left(\frac{p_s}{q_s} - 1\right)(x)\nabla \cdot \left[a(s,x)\nabla\left[\frac{p_s}{q_s}\right](x)q_s(x)\right] dxds$$

where $\widetilde{sign}(r) = -\mathbf{1}_{(-\infty,0)}(r) + \mathbf{1}_{(0,\infty)}(r)$.

Recall:

$$\frac{d}{dt}H_U(p_t|p_\infty) = -\frac{1}{2}I_U(p_t|p_\infty)$$

where U-Fischer information is $I_U(p_t|p_\infty) =$

$$\int_{\{\frac{p_t}{p_\infty}(x)>0\}} U''\left(\frac{p_t}{p_\infty}(x)\right)\left(\nabla^*\left[\frac{p_t}{p_\infty}\right]a(t,\cdot)\nabla\left[\frac{p_t}{p_\infty}\right]\right)(x)p_\infty(x)dx.$$

Bakry-Emery approach: provide conditions for

$$\frac{d}{dt}I_U(p_t|p_\infty) \leq -2\lambda I_U(p_t|p_\infty)$$

to hold for some $\lambda > 0$. Then, $\forall s \geq 0$, $I_U(p_s|p_\infty) \leq e^{-2\lambda s}I_U(p_0|p_\infty)$ and

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Framework

*H*4) Regular time-homogeneous coefficients $\sigma(x)$ and $b(x)$.

$H5)_{p_\infty}$ $Q_0(dx) = p_\infty(x)dx$, $p_\infty(x) > 0$ invariant, regular. $\mathbb{P}_\infty^{T\to 0} := \mathbb{Q}^{T\to 0}$

In (a possible enlargment of) $\mathbb{P}_\infty^{T\to 0}$ there is a Brownian motion $(\bar{W}_t)_{t\in[0,T]}$ such that

$$dY_t = \bar{b}(Y_t)dt + \sigma(Y_t)d\bar{W}_t, \quad t \in [0, T]$$

$$\text{with } \bar{b}_i(y) = -b_i(y) + \frac{\partial_j(a_{ij}(y)p_\infty(y))}{p_\infty(y)}.$$

Trajectorial uniqueness holds for this SDE $\Rightarrow \mathbb{P}_\infty^{T\to 0}$ unique and therefore extremal solution of (MP_{Q_0})

$H6)_{p_0}$ Regularity of $p_t(x)$

*H*7) The convex function $U : [0, \infty) \to \mathbb{R}$ is of class C^4 on $(0, +\infty)$, continuous on $[0, +\infty)$ and satisfies $U(1) = U'(1) = 0$.

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Proposition

Let U_δ of class C^4 in $\mathbb{R}$ such that $U_\delta(r) = U(r+\delta), r \geq 0$.
Let $\rho_t = p_{T-t}/p_\infty$ and all functions be computed at (t, Y_t). Then

$$d\left[U''_\delta(\rho_t)\nabla^*\rho_t a \nabla \rho_t\right](Y_t) = tr(\Lambda_\delta \Gamma)dt + U''_\delta(\rho)\bar{\theta}dt + d\hat{M}^{(\delta)}$$

where

$$d\hat{M}^{(\delta)} := \partial_k \left[U''_\delta(\rho)\nabla^*\rho a \nabla \rho\right]\sigma_{kr} d\bar{W}^r,$$

$\mathcal{G}_t - \mathbb{Q}^{T\to 0}$local martingale, Λ_δ and Γ square matrices defined by

$$\Lambda_\delta := \begin{bmatrix} U''_\delta(\rho) & U^{(3)}_\delta(\rho) \\ U^{(3)}_\delta(\rho) & \frac{1}{2}U^{(4)}_\delta(\rho) \end{bmatrix}$$

$$\Gamma := \begin{bmatrix} \nabla^*(\sigma_{\bullet i}\cdot\nabla\rho)a\nabla(\sigma_{\bullet i}\cdot\nabla\rho) & (\sigma_{\bullet i}\cdot\nabla\rho)\nabla^*\rho\, a\nabla(\sigma_{\bullet i}\cdot\nabla\rho) \\ (\sigma_{\bullet i}\cdot\nabla\rho)\nabla^*\rho\, a\nabla(\sigma_{\bullet i}\cdot\nabla\rho) & |\nabla^*\rho a \nabla\rho|^2 \end{bmatrix}.$$

and

$$\bar{\theta} = 2\Big\{ [\sigma_{l'i}\partial_{l'}\rho a_{mk}\partial_m\sigma_{li}\partial_{lk}\rho] + \sigma_{l'i}\partial_{l'}\rho\partial_l\rho\left[\bar{b}_m\partial_m\sigma_{li} + \frac{1}{2}a_{mk}\partial_{mk}\sigma_{li}\right]$$
$$- a_{ll'}\partial_{l'}\rho\left[\sigma_{kr}\partial_{kj}\rho\partial_l\sigma_{jr} + \partial_k\rho\partial_l\bar{b}_k\right]\Big\}$$

Sketch: stochastic flow

$$d\xi_t^i(x) = \sigma_{ik}(\xi_t(x))d\bar{W}_t^k + \bar{b}_i(\xi_t(x))dt, \quad (t,x) \in [0,T) \times \mathbb{R}^d, \ i = 1, \dots d,$$

and $\xi_0(x) = x$ (so that $\xi_t(Y_0) = Y_t$).
The continuous $\mathcal{G}_t - \mathbb{P}_\infty^{T\to 0} -$ local martingales $(D_t(x) : t \in [0,T])_{x\in\mathbb{R}^d}$

$$dD_t(x) := [\sigma_{ik}\partial_i\rho]\,(t, \xi_t(x))d\bar{W}_t^k \quad , \quad D_0(x) = \frac{p_T}{p_\infty}(x) = \rho_0(x).$$

satisfy $D_t(x) = \rho_t(\xi_t(x))$. Then
$\nabla\rho_t(\xi_t(x)) = (\nabla_x\xi_t(x))^{-1}\nabla_x[\rho_t(\xi_t(x))]$, use the previous and SDE for $d(\nabla_x\xi_t(x))^{-1}$ in the Itô product rule.

Itô formula for $U_\delta(\rho_t)$.

For $\nabla^*\rho_t a\nabla\rho_t$ use product rule like this: $(\sigma^*\nabla\rho_t).(\sigma^*\nabla\rho_t)$. Why?

Cauchy Schwarz $\Rightarrow$

$$
\begin{aligned}
((\sigma_{\bullet i} \cdot \nabla\rho)\nabla^*\rho\, a\nabla(\sigma_{\bullet i} \cdot \nabla\rho))^2 &= ((\sigma_{\bullet i} \cdot \nabla\rho)\sigma^*\nabla\rho.\sigma^*\nabla(\sigma_{\bullet i} \cdot \nabla\rho))^2 \\
&\leq \sum_i (\sigma_{\bullet i} \cdot \nabla\rho)^2 |\sigma^*\nabla\rho|^2 \sum_i |\sigma^*\nabla(\sigma_{\bullet i} \cdot \nabla\rho)|^2 \\
&= |\nabla^*\rho a\nabla\rho|^2 \times \nabla^*(\sigma_{\bullet i} \cdot \nabla\rho) a\nabla(\sigma_{\bullet i} \cdot \nabla\rho).
\end{aligned}
$$

the determinant of the matrix Γ is nonnegative, and the matrix is positive semidefinite. From now on we assume

$$
\forall r \in (0,\infty),\ (U^{(3)}(r))^2 \leq \frac{1}{2} U''(r)U^{(4)}(r) \text{ (e.g. } r\ln(r) \text{ and } (r-1)^2)
$$

which ensures that Λ is also positive semidefinite.

Corollary $d\left[U''_\delta(\rho)\nabla^*\rho a\nabla\rho\right] \geq U''_\delta(\rho)\bar{\theta}dt + d\hat{M}^{(\delta)}$.

THEOREM 2

Let $\Theta \in \mathbb{R}^{d\otimes d}$ be defined by

$$\Theta_{ll'} = \sigma_{l'i}[\bar{b}_m \partial_m \sigma_{li} + \frac{1}{2} a_{mk} \partial_{mk} \sigma_{li}] - a_{kl'} \partial_k \bar{b}_l + (\sigma_{ki} a_{ml'} - \sigma_{l'i} a_{mk}) \partial_m \sigma_{li} \partial_k \ln(p_\infty) + \partial_k [(\sigma_{ki} a_{ml'} - \sigma_{l'i} a_{mk}) \partial_m \sigma_{li}].$$

and assume some integrability w.r.t. p_∞ of ρ_t and its derivatives (for instance for nice p_0) Then, for a.e. $t \in [0, T]$ one has

$$\frac{d}{dt} \underbrace{\int_{\rho_t > 0} U''(\rho_t)[\nabla^* \rho_t a \nabla \rho_t] p_\infty dx}_{I_U(p_{T-t}|p_\infty)} \geq \int_{\rho_t > 0} U''(\rho_t) \nabla^* \rho_t (\Theta + \Theta^*) \nabla \rho_t p_\infty dx.$$

Moreover, if $H_U(p_s|p_\infty)$ is finite for some $s \geq 0$, the symmetric matrix $(\Theta + \Theta^*)(t, x)$ is $p_\infty(x)dxdt$ a.e. positive semidefinite and the diffusion matrix a is locally uniformly strictly positive definite, or hypoellipticity conditions hold, then $H_U(p_t|p_\infty) \to 0$ as $t \to \infty$.

heart of the proof:

$$[U_\delta''(\rho)\nabla^*\rho a\nabla\rho](t,Y_t) - [U_\delta''(\rho)\nabla^*\rho a\nabla\rho](r,Y_r)$$

$$\geq \hat{M}_t^{(\delta)} - \hat{M}_r^{(\delta)} + 2\int_r^t U_\delta''(\rho)\left[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}\right]\partial_{l'}\rho\partial_m\sigma_{li}\partial_{kl}\rho ds$$

$$+2\int_r^t U_\delta''(\rho)\partial_{l'}\rho\partial_l\rho\left(\sigma_{l'i}\left[\bar{b}_m\partial_m\sigma_{li} + \frac{1}{2}a_{mk}\partial_{mk}\sigma_{li}\right] - a_{ml'}\partial_m\bar{b}_l\right) ds.$$

One has $U_\delta''(\rho)\left[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}\right]\partial_{l'}\rho\partial_m\sigma_{li}\partial_{kl}\rho =$

$$\frac{1}{p_\infty}\partial_k\left(\partial_l\rho\partial_{l'}\rho U_\delta''(\rho)\left[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}\right]\partial_m\sigma_{li}\, p_\infty\right)$$

$$-\partial_l\rho\partial_{l'}\rho U_\delta''(\rho)\left[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}\right]\partial_m\sigma_{li}\partial_k \ln p_\infty$$

$$-\partial_l\rho\partial_{l'}\rho U_\delta''(\rho)\partial_k\left(\left[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}\right]\partial_m\sigma_{li}\right)$$

since

$$\partial_{kl'}\rho U_\delta''(\rho)\left[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}\right] = 0$$

and

$$\partial_k(U_\delta''(\rho))\partial_{l'}\rho\left[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}\right] = U_\delta^{(3)}(\rho)\partial_k\rho\partial_{l'}\rho\left[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}\right] = 0.$$

THEOREM 3

If the matrix Θ satisfies the non-intrinsic Bakry-Emery criterion

$$\textit{NIBEC)} \quad \exists \lambda > 0,\ \forall x \in \mathbb{R}^d,\ (\Theta + \Theta^*)(x) \geq \lambda a(x)$$

then $\frac{d}{dt} I_U(p_t|p_\infty) \leq -\lambda I_U(p_t|p_\infty)$ and the non-increasing function $t \mapsto H_U(p_t|p_\infty)$ converges at exponential rate λ to its limit as $t \to \infty$.

When, moreover, the diffusion matrix a is locally uniformly strictly positive definite or hypoellipticity conditions hold, then this limit is equal to 0 as soon as $H_U(p_s|p_\infty)$ is finite for some $s \geq 0$, and the convex Sobolev inequality

$$H_U(p|p_\infty) \leq \frac{1}{\lambda} I_U(p|p_\infty)$$

holds for any probability density p on $\mathbb{R}^d$.

Remark

i) $\frac{1}{2}(\Theta + \Theta^*) =$

$$\frac{1}{2}\bar{b}_m \partial_m a_{ll'} - \frac{1}{2}(a_{kl'}\partial_k \bar{b}_l + a_{kl}\partial_k \bar{b}_{l'}) + \frac{1}{4} a_{mk}\partial_{mk} a_{ll'} - \frac{1}{2} a_{mk}\partial_m \sigma_{li}\partial_k \sigma_{l'i}$$
$$+ \frac{1}{2}\sigma_{ki}(\partial_m \sigma_{li} a_{ml'} + \partial_m \sigma_{l'i} a_{ml})\partial_k \ln(p_\infty) - \frac{1}{2} a_{mk}\partial_m a_{ll'}\partial_k \ln(p_\infty)$$
$$+ \frac{1}{2}\partial_k[\sigma_{ki}(\partial_m \sigma_{li} a_{ml'} + \partial_m \sigma_{l'i} a_{ml}) - a_{mk}\partial_m a_{ll'}]$$

non-intrisic expression that cannot be rewritten without making use of the square root σ.

ii) In case $a = 2\nu I_d$ and $b = -(\nabla V + F)$ with F such that $\nabla.(e^{-V/\nu} F) = 0$, then $p_\infty \propto e^{-V/\nu}$, $\bar{b} = -b + 2\nu \nabla \ln p_\infty = -\nabla V + F$ and $\Theta = 2\nu(\nabla^2 V - \nabla F)$. **If we chose** $\sigma = \sqrt{2\nu} I_d$, then *NIBEC)* writes $\exists \lambda > 0, \ \forall x \in \mathbb{R}^d, \ \nabla^2 V(x) - \frac{\nabla F + \nabla F^*}{2}(x) \geq \lambda I_d$ which is exactly condition (A2) of Arnold, Carlen and Ju (2008) .

Remark

i) $\frac{1}{2}(\Theta + \Theta^*) =$

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$$+ \frac{1}{2}\sigma_{ki}(\partial_m \sigma_{li} a_{ml'} + \partial_m \sigma_{l'i} a_{ml})\partial_k \ln(p_\infty) - \frac{1}{2} a_{mk}\partial_m a_{ll'}\partial_k \ln(p_\infty)$$
$$+ \frac{1}{2}\partial_k[\sigma_{ki}(\partial_m \sigma_{li} a_{ml'} + \partial_m \sigma_{l'i} a_{ml}) - a_{mk}\partial_m a_{ll'}]$$

non-intrisic expression that cannot be rewritten without making use of the square root σ.

ii) In case $a = 2\nu I_d$ and $b = -(\nabla V + F)$ with F such that $\nabla.(e^{-V/\nu} F) = 0$, then $p_\infty \propto e^{-V/\nu}$, $\bar{b} = -b + 2\nu\nabla \ln p_\infty = -\nabla V + F$ and $\Theta = 2\nu(\nabla^2 V - \nabla F)$. **If we chose** $\sigma = \sqrt{2\nu} I_d$, then *NIBEC)* writes $\exists\lambda > 0,\ \forall x \in \mathbb{R}^d,\ \nabla^2 V(x) - \frac{\nabla F + \nabla F^*}{2}(x) \geq \lambda I_d$ which is exactly condition (A2) of Arnold, Carlen and Ju (2008) .

Application: choice of σ can provide convergence results when classic BEC) fails, without modifying the PDE

$d = 2$ and for each $(x_1, x_2) \in \mathbb{R}^2$,

$$a(x_1, x_2) = I_2, \quad \text{and} \quad b(x_1, x_2) = -\nabla V(x_1, x_2)$$

with V convex C^2 potential

$$V(x_1, x_2) := |x_1|^2 + |x_1 - x_2|^{2+\alpha} + |x_2|^{2+\alpha}$$

for some $\alpha \in (0, 1)$. Invariant distribution : $p_\infty \propto e^{-2V}$

$$\begin{aligned}\partial_1 V =& 2x_1 + (2+\alpha) sign(x_1 - x_2)|x_1 - x_2|^{1+\alpha} \\ \partial_2 V =& (2+\alpha) sign(x_2)|x_2|^{1+\alpha} \\ &+ (2+\alpha) sign(x_2 - x_1)|x_2 - x_1|^{1+\alpha}\end{aligned}$$

and

$$\nabla^2 V = \begin{pmatrix} 2 & 0 \\ 0 & (2+\alpha)(1+\alpha)|x_2|^\alpha \end{pmatrix} + (2+\alpha)(1+\alpha)|x_1 - x_2|^\alpha \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Classic Bakry-Emery criterion fails since $\nabla^2 V(0,0)$ is singular. Smallest eigenvalue of $\nabla^2 V(x_1, x_2)$, is given by

$$\gamma_-(x_1, x_2) = 1 + \kappa_1 + \kappa_2/2 - \sqrt{1 + \kappa_1^2 - \kappa_2 + \kappa_2^2/4} \geq 0$$

with $\kappa_1(x_1, x_2) := (2+\alpha)(1+\alpha)|x_1 - x_2|^\alpha$ and $\kappa_2(x_1, x_2) := (2+\alpha)(1+\alpha)|x_2|^\alpha$ so that

$$\text{for } \varepsilon > 0 \text{ small,} \quad \inf_{|x_1| \vee |x_2| \geq \varepsilon} \gamma_-(x_1, x_2) \geq (2+\alpha)(1+\alpha)\varepsilon^\alpha + o(\varepsilon^\alpha).$$

Square root σ of the identity matrix of the form

$$\sigma(x_1, x_2) = \begin{pmatrix} \cos\phi(x_1, x_2) & \sin\phi(x_1, x_2) \\ -\sin\phi(x_1, x_2) & \cos\phi(x_1, x_2) \end{pmatrix}$$

for a function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ of class C^2. Then

$$\frac{1}{2}(\Theta + \Theta^*) = \nabla^2 V + \begin{pmatrix} \partial_{12}\phi & \frac{\partial_{22}\phi - \partial_{11}\phi}{2} \\ \frac{\partial_{22}\phi - \partial_{11}\phi}{2} & -\partial_{12}\phi \end{pmatrix}$$
$$- \frac{1}{2}|\nabla\phi|^2 I_2 + \begin{pmatrix} -2\partial_1\phi\partial_2 V & \partial_1\phi\partial_1 V - \partial_2\phi\partial_2 V \\ \partial_1\phi\partial_1 V - \partial_2\phi\partial_2 V & 2\partial_2\phi\partial_1 V \end{pmatrix}$$

For $\varepsilon > 0$ to be chosen small and a C^2 function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(s) = s$ if $|s| \leq 1$ and $\varphi(s) = 0$ if $|s| \geq 2$ we define

$$\phi(x_1, x_2) = -\varepsilon\varphi_\varepsilon(x_1)\varphi_\varepsilon(x_2), \quad (x_1, x_2) \in \mathbb{R}^2$$

where $\varphi_\varepsilon(s) = \varepsilon\varphi(s/\varepsilon)$. Notice that

$$\varphi_\varepsilon = O(\varepsilon), \quad \varphi''_\varepsilon = O(1/\varepsilon), \quad \text{and } \varphi'_\varepsilon = \begin{cases} 1 & \text{if } |s| \leq \varepsilon, \\ O(1) & \text{if } \varepsilon < |s| < 2\varepsilon, \\ 0 & \text{if } |s| \geq 2\varepsilon. \end{cases}$$

Let $B_\varepsilon := \{(x_1, x_2) \in \mathbb{R}^2 \; s.t. \, |x_1| \vee |x_2| \leq \varepsilon\}$ and $C_\varepsilon := B_{2\varepsilon} \backslash B_\varepsilon$.

$$\text{On } B^c_{2\varepsilon}, \; \phi \equiv 0 \text{ and } \frac{1}{2}(\Theta + \Theta^*) = \nabla^2 V \geq \inf_{|x_1|\vee|x_2|\geq 2\varepsilon} \gamma_-(x_1, x_2) I_2.$$

On $B_{2\varepsilon}$,

$$\partial_1\phi = O(\varepsilon^2),\ \partial_2\phi = O(\varepsilon^2),\ \partial_1 V = O(\varepsilon) \text{ and } \partial_2 V = O(\varepsilon^{1+\alpha}).$$

Therefore

$$-\frac{1}{2}|\nabla\phi|^2 I_2 + \begin{pmatrix} -2\partial_1\phi\partial_2 V & \partial_1\phi\partial_1 V - \partial_2\phi\partial_2 V \\ \partial_1\phi\partial_1 V - \partial_2\phi\partial_2 V & 2\partial_2\phi\partial_1 V \end{pmatrix} = O(\varepsilon^3).$$

$$\begin{pmatrix} \partial_{12}\phi & \frac{\partial_{22}\phi-\partial_{11}\phi}{2} \\ \frac{\partial_{22}\phi-\partial_{11}\phi}{2} & -\partial_{12}\phi \end{pmatrix} = \begin{cases} \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \text{ on } B_\varepsilon \\ O(\varepsilon) \text{ on } C_\varepsilon \end{cases}$$

And $\frac{1}{2}(\Theta + \Theta^*) =$

$$\begin{cases} \nabla^2 V + \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} + O(\varepsilon^3) \geq \begin{pmatrix} 2-\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} + O(\varepsilon^3) \text{ on } B_\varepsilon \\ \nabla^2 V + O(\varepsilon) \geq (2+\alpha)(1+\alpha)\varepsilon^\alpha I_2 + o(\varepsilon^\alpha) \text{ on } C_\varepsilon \end{cases}$$

Thus **NIBEC)** holds for sufficiently small $\varepsilon > 0$.

Questions and future work:

- (differential) geometrical meaning of NIBEC?
- optimal choice of the square root?
- a non "ε type" example.
- is there an interpretation in terms of coupling?
- what about the general case $q_0 \neq p_\infty$?

Thank you!