Optimal growth for linear processes with affine control
Application to a protein amplification technique

Pierre Gabriel (Inria Lyon, France)

Joint work with Vincent Calvez

10th ICOR, La Habana, March 7th, 2012

The growth-fragmentation equation

$$\partial_t f(t,\xi) + \partial_\xi\big(\tau(\xi) f(t,\xi)\big) = \mathcal{F} f(t,\xi) \qquad t,\, \xi > 0$$

The growth-fragmentation equation

$$\partial_t f(t,\xi) + \partial_\xi\big(\tau(\xi) f(t,\xi)\big) = \mathcal{F} f(t,\xi) \qquad t,\, \xi > 0$$

Dynamics of a structured population submitted to growth and fragmentation

$\rightarrow$ *Example:* $f(t,\xi)$ *= density of polymers of size* ξ *at time* t

The growth-fragmentation equation

$$\partial_t f(t,\xi) + \partial_\xi\big(\tau(\xi) f(t,\xi)\big) = \mathcal{F} f(t,\xi) \qquad t,\, \xi > 0$$

Dynamics of a structured population submitted to growth and fragmentation

→ *Example:* $f(t,\xi)$ = *density of polymers of size* ξ *at time* t

$\tau(\xi)$: growth rate

$\mathcal{F}$: fragmentation operator

$$\mathcal{F} f(\xi) := 2\int_\xi^\infty \beta(\zeta)\kappa(\xi,\zeta) f(\zeta)\, d\zeta - \beta(\xi) f(\xi)$$

This operator is mass preserving: $\displaystyle\int_0^\infty \xi\, \mathcal{F} f(\xi)\, d\xi = 0.$

Protein Misfolded Cyclic Amplification

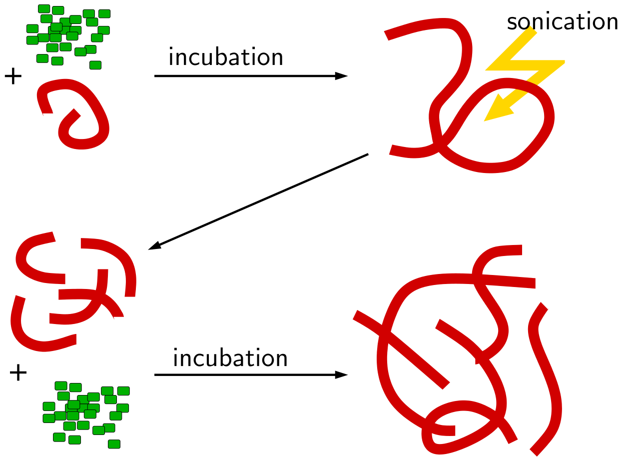


Figure: The PMCA principle.

Modelling the PMCA

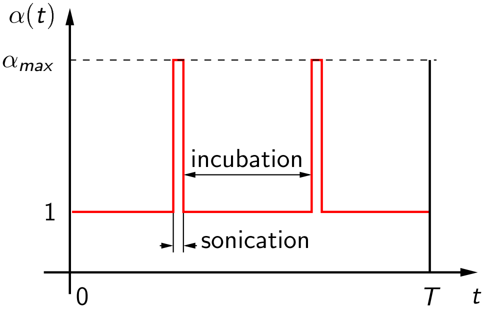
$$\partial_t f(t,\xi) + \partial_\xi\big(\tau(\xi) f(t,\xi)\big) = \alpha(t)\,\mathcal{F}\,f(t,\xi)$$

$\alpha(t)$: sonication parameter

Modelling the PMCA

$$\partial_t f(t,\xi) + \partial_\xi\big(\tau(\xi) f(t,\xi)\big) = \alpha(t)\,\mathcal{F}\,f(t,\xi)$$

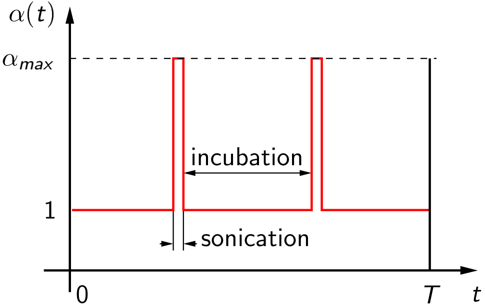
$\alpha(t)$: sonication parameter



Modelling the PMCA

$$\partial_t f(t,\xi) + \partial_\xi\big(\tau(\xi) f(t,\xi)\big) = \alpha(t)\,\mathcal{F} f(t,\xi)$$

$\alpha(t)$: sonication parameter



Problem: maximize the quantity $\int_0^\infty \xi f(T,\xi)\,d\xi$ for a given final time T.

Discrete model

$$\begin{cases} \dot{x}_\alpha(t) = \big(G + \alpha F\big) x_\alpha(t), \\ x_\alpha(0) = x. \end{cases}$$

$$G = \begin{pmatrix} -\tau_1 & 0 & 0 \\ \tau_1 & -\tau_2 & 0 \\ 0 & \tau_2 & 0 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & 2\beta_2 & \beta_3 \\ 0 & -\beta_2 & \beta_3 \\ 0 & 0 & -\beta_3 \end{pmatrix}$$

The final reward writes $\langle m, x(T) \rangle$ with $m = (1\ 2\ 3)^T$ the size vector which satisfies $m^T F = 0$.

Particular strategy: Constant control

We investigate the asymptotic behavior of the optimal reward when $T \to +\infty$.

For a constant control α, we know from the Perron-Frobenius theory that

$$r_\alpha(T,x) := \frac{1}{T} \log \langle m, x_\alpha(T) \rangle \xrightarrow[T\to\infty]{} \lambda_P(\alpha).$$

Particular strategy: Constant control

We investigate the asymptotic behavior of the optimal reward when $T \to +\infty$.

For a constant control α, we know from the Perron-Frobenius theory that

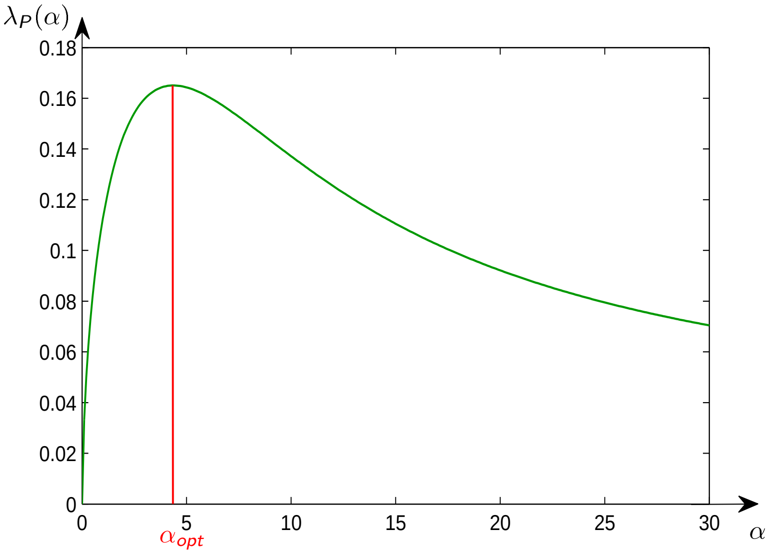
$$r_\alpha(T,x) := \frac{1}{T}\log\langle m, x_\alpha(T)\rangle \xrightarrow[T\to\infty]{} \lambda_P(\alpha).$$

Proposition

If $\tau_2 > 2\,\tau_1$, then $\exists\,\alpha_{opt} > 0$ such that

$$\forall\,\alpha > 0\,, \qquad \lambda_P(\alpha) \leqslant \lambda_P(\alpha_{opt})\,.$$

Optimal eigenvalue



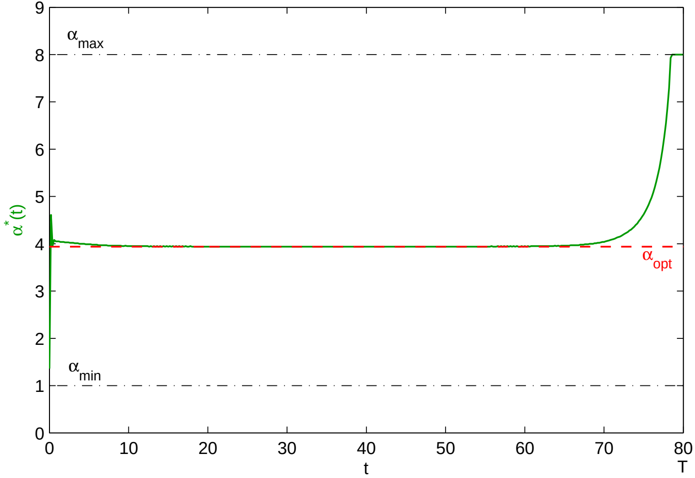
Optimal control

Assume that there exists an optimum α_{opt} for λ_P and consider

$$0 < \alpha_{min} < \alpha_{opt} < \alpha_{max}.$$

Problem: find a measurable control $\alpha^* : (0, T) \to [\alpha_{min}, \alpha_{max}]$ which maximizes the reward $\langle m, x_\alpha(T) \rangle$.

Optimal control and the best Perron eigenvalue



The value function and the HJB equation

Define the value function

$$v(T, x) = \sup_{\alpha} \langle m, x_\alpha(T) \rangle$$

where the supremum is taken over all measurable functions $\alpha : (0, T) \to [\alpha_{min}, \alpha_{max}]$.

The value function and the HJB equation

Define the value function

$$v(T, x) = \sup_{\alpha} \langle m, x_\alpha(T) \rangle$$

where the supremum is taken over all measurable functions $\alpha : (0, T) \to [\alpha_{min}, \alpha_{max}]$.

It satisfies the Hamilton-Jacobi-Bellman equation

$$\partial_t v(t, x) - \max_{a \in [\alpha_{min}, \alpha_{max}]} \langle (G + aF)x, \nabla_x v(t, x) \rangle = 0.$$

The value function and the HJB equation

Define the value function

$$v(T, x) = \sup_{\alpha} \langle m, x_\alpha(T) \rangle$$

where the supremum is taken over all measurable functions $\alpha : (0, T) \to [\alpha_{min}, \alpha_{max}]$.

It satisfies the Hamilton-Jacobi-Bellman equation

$$\partial_t v(t, x) - \max_{a \in [\alpha_{min}, \alpha_{max}]} \langle (G + aF)x, \nabla_x v(t, x) \rangle = 0.$$

We are interested in the long-time behavior $\lim_{T \to +\infty} v(T, x)$.

Reduction to a compact space / linear growth

We perform a logarithmic change of variable $u(t,x) = \log v(t,x)$.

Reduction to a compact space / linear growth

We perform a logarithmic change of variable $u(t,x) = \log v(t,x)$.

The reward function satisfies

$$\frac{d}{dt}\log\langle m, x_\alpha(t)\rangle = \frac{\langle m, \dot{x}_\alpha(t)\rangle}{\langle m, x_\alpha(t)\rangle} = \langle m, (G + \alpha(t)F)y_\alpha(t)\rangle = \langle m, G\, y_\alpha(t)\rangle$$

where $y_\alpha = \frac{x_\alpha}{\langle m, x_\alpha\rangle}$ is the projection on the simplex

$$\mathcal{S} = \{y \geqslant 0 \;:\; \langle m, y\rangle = 1\}.$$

Reduction to a compact space / linear growth

We perform a logarithmic change of variable $u(t,x) = \log v(t,x)$.

The reward function satisfies

$$\frac{d}{dt}\log\langle m, x_\alpha(t)\rangle = \frac{\langle m, \dot{x}_\alpha(t)\rangle}{\langle m, x_\alpha(t)\rangle} = \langle m, (G + \alpha(t)F)y_\alpha(t)\rangle = \langle m, G\, y_\alpha(t)\rangle$$

where $y_\alpha = \frac{x_\alpha}{\langle m, x_\alpha\rangle}$ is the projection on the simplex

$$\mathcal{S} = \{y \geqslant 0 \ : \ \langle m, y\rangle = 1\}.$$

Then we have for the logarithmic value function

$$u(t,x) = \sup_\alpha\{\log\langle m, x_\alpha(t)\rangle\} = \sup_\alpha\left\{\int_0^t L(y_\alpha(s))\, ds\right\} + \log\langle m, x\rangle.$$

where $L(y) = \langle m, G\, y\rangle$.

Reduction to a compact space / linear growth

We can write a closed equation for y_α

$$\begin{aligned}\dot{y}_\alpha(t) &= \frac{\dot{x}_\alpha}{\langle m, x_\alpha(t)\rangle} - \frac{\langle m, \dot{x}_\alpha(t)\rangle}{\langle m, x_\alpha(t)\rangle}\frac{x_\alpha(t)}{\langle m, x_\alpha(t)\rangle} \\ &= (G + \alpha(t)F)y_\alpha(t) - \langle m, G\, y_\alpha(t)\rangle\, y_\alpha(t).\end{aligned}$$

Reduction to a compact space / linear growth

We can write a closed equation for y_α

$$\begin{aligned}\dot{y}_\alpha(t) &= \frac{\dot{x}_\alpha}{\langle m, x_\alpha(t)\rangle} - \frac{\langle m, \dot{x}_\alpha(t)\rangle}{\langle m, x_\alpha(t)\rangle}\frac{x_\alpha(t)}{\langle m, x_\alpha(t)\rangle} \\ &= (G + \alpha(t)F)y_\alpha(t) - \langle m, G\, y_\alpha(t)\rangle\, y_\alpha(t).\end{aligned}$$

Denoting

$$b(y, \alpha) = (G + \alpha F)y - \langle m, G\, y\rangle\, y$$

we finally obtain a problem reduced to a compact set (the simplex $\mathcal{S}$)

$$u(t, y) = \sup_\alpha \left\{ \int_0^t L(y_\alpha(s))\, ds \ : \ \dot{y}_\alpha(s) = b(y_\alpha(s), \alpha(s)),\ y_\alpha(0) = y \right\}.$$

Reduction to a compact space / linear growth

We can write a closed equation for y_α

$$\dot{y}_\alpha(t) = \frac{\dot{x}_\alpha}{\langle m, x_\alpha(t)\rangle} - \frac{\langle m, \dot{x}_\alpha(t)\rangle}{\langle m, x_\alpha(t)\rangle}\frac{x_\alpha(t)}{\langle m, x_\alpha(t)\rangle}$$
$$= (G + \alpha(t)F)y_\alpha(t) - \langle m, G\, y_\alpha(t)\rangle\, y_\alpha(t).$$

Denoting

$$b(y, \alpha) = (G + \alpha F)y - \langle m, G\, y\rangle\, y$$

we finally obtain a problem reduced to a compact set (the simplex $\mathcal{S}$)

$$u(t, y) = \sup_\alpha \left\{ \int_0^t L(y_\alpha(s))\, ds \;:\; \dot{y}_\alpha(s) = b(y_\alpha(s), \alpha(s)),\; y_\alpha(0) = y \right\}.$$

This function satisfies the HJB equation

$$\partial_t u(t, y) + H(y, \nabla_y u(t, y)) = 0$$

where the Hamiltonian is $H(y, p) = -\max_\alpha\{\langle b(y, \alpha), p\rangle + L(y)\}$.

Result

Define the infinite horizon problem

$$u_\epsilon(y) = \sup_\alpha \left\{ \int_0^\infty e^{-\epsilon s} L(y_\alpha(s))\, ds \;\;:\;\; \dot{y}_\alpha(s) = b(y_\alpha(s), \alpha(s)),\;\; y_\alpha(0) = y \right\}.$$

This function satisfies the stationary HJB equation

$$\epsilon u_\epsilon(y) + H(y, D_y u_\epsilon(y)) = 0.$$

Result

Define the infinite horizon problem

$$u_\epsilon(y) = \sup_\alpha \left\{ \int_0^\infty e^{-\epsilon s} L(y_\alpha(s))\, ds \ : \ \dot{y}_\alpha(s) = b(y_\alpha(s), \alpha(s)),\ y_\alpha(0) = y \right\}.$$

This function satisfies the stationary HJB equation

$$\epsilon u_\epsilon(y) + H(y, D_y u_\epsilon(y)) = 0.$$

Theorem

Under technical assumptions, there exists a constant λ_{HJ} such that

$$\epsilon u_\epsilon(y) \xrightarrow[\epsilon \to 0]{} \lambda_{HJ} \qquad \text{uniformly in } y \in \mathcal{S}.$$

Result

Define the infinite horizon problem

$$u_\epsilon(y) = \sup_\alpha \left\{ \int_0^\infty e^{-\epsilon s} L(y_\alpha(s))\, ds \;:\; \dot{y}_\alpha(s) = b(y_\alpha(s), \alpha(s)),\; y_\alpha(0) = y \right\}.$$

This function satisfies the stationary HJB equation

$$\epsilon u_\epsilon(y) + H(y, D_y u_\epsilon(y)) = 0.$$

Theorem

Under technical assumptions, there exists a constant λ_{HJ} such that

$$\epsilon u_\epsilon(y) \xrightarrow[\epsilon \to 0]{} \lambda_{HJ} \qquad \text{uniformly in } y \in \mathcal{S}.$$

Corollary

The value function $u(t, y)$ satisfies the ergodic property

$$\frac{1}{T} u(T, y) \xrightarrow[T \to +\infty]{} \lambda_{HJ} \qquad \text{uniformly in } y \in \mathcal{S}.$$

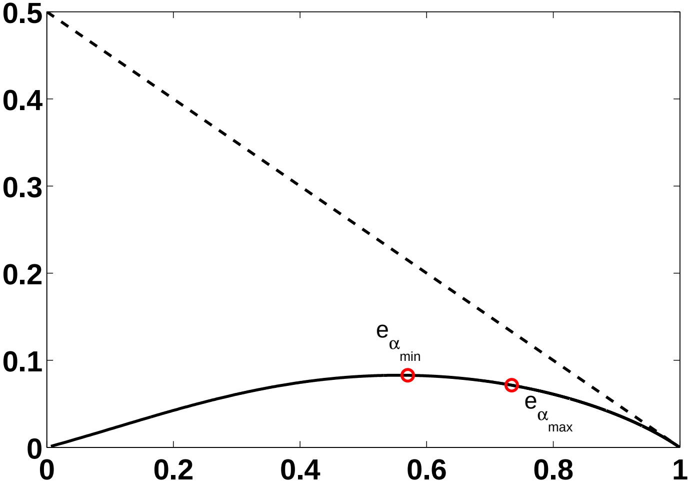
Sketch of the proof

We mainly follow
Arisawa, Ergodic problem for the Hamilton-Jacobi-Bellman equation,
Ann. Inst. Henri poincaré (C) Nonlin. Anal. (1998).

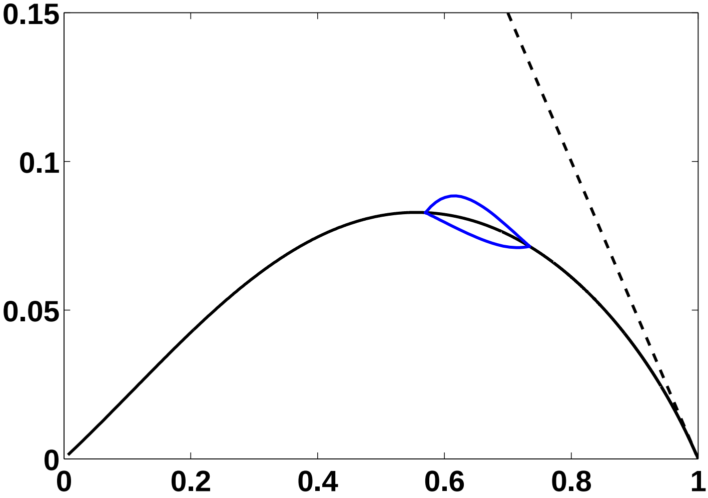
Outline:

- identify the so-called ergodic set $\mathcal{Z}_0 \subset \mathcal{S}$
- demonstrate its controllability: this property enables to prove the convergence to λ_{HJ} when $y \in \mathcal{Z}_0$
- demonstrate its attractiveness: this property enables to extend the convergence to any $y \in \mathcal{S}$

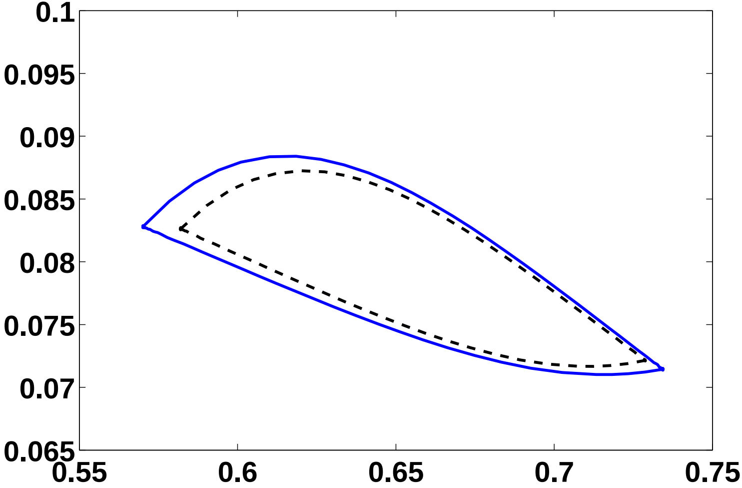
The line of eigenvectors



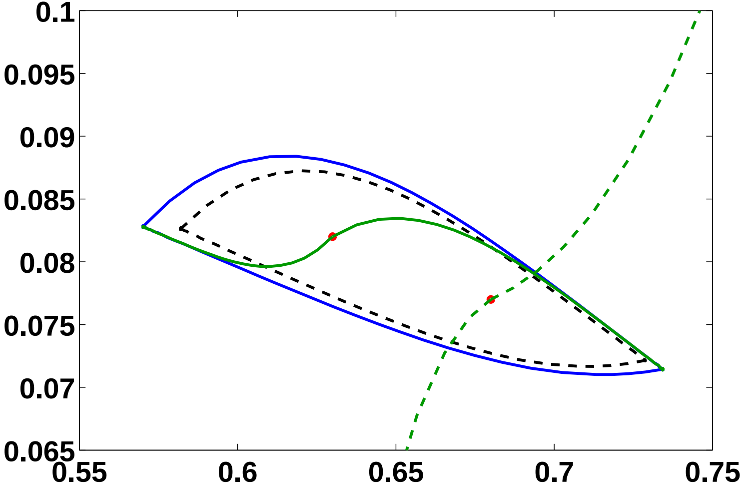
The ergodic set $\mathcal{Z}_0$



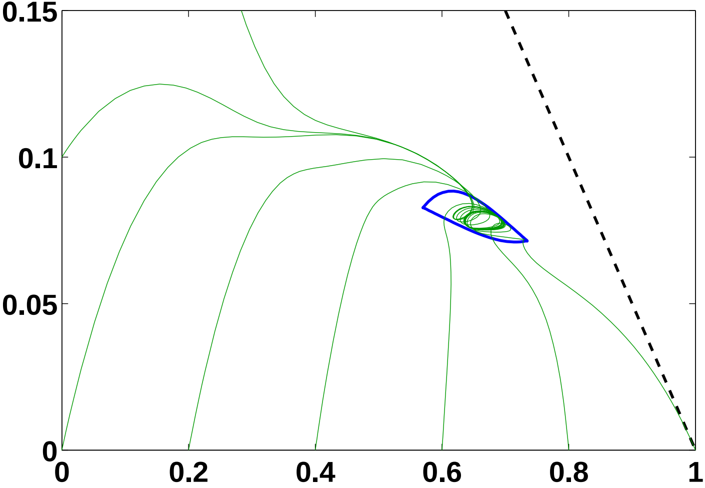
The ergodic set minus a narrow band



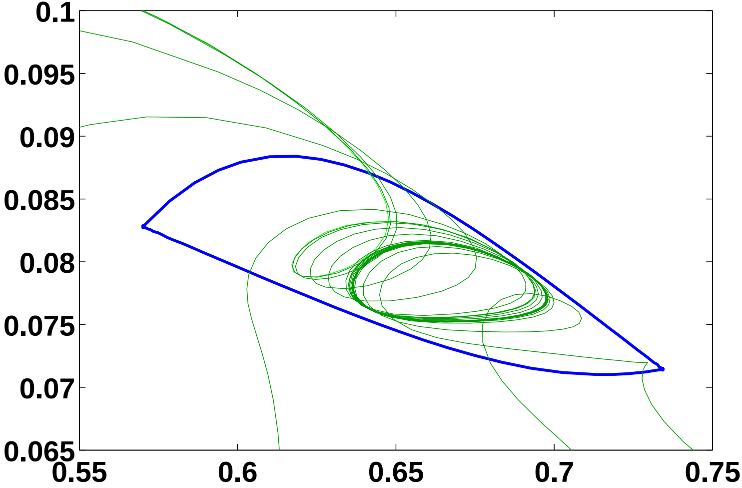
The bang-bang controllability



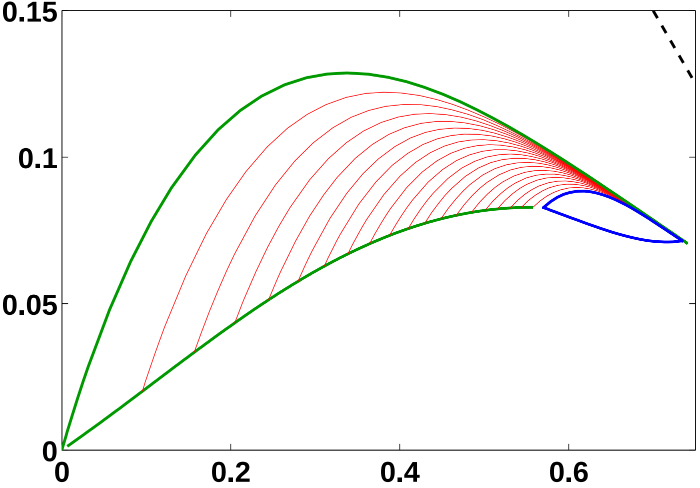
Attractiveness of $\mathcal{Z}_0$



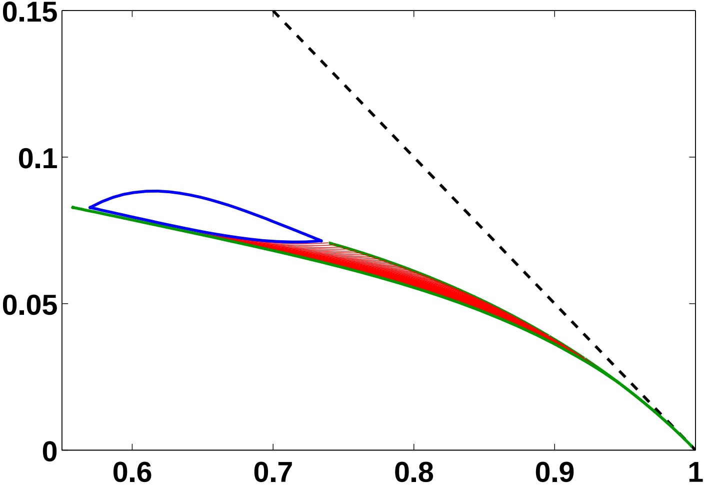
Attractiveness of $\mathcal{Z}_0$



Tunnelling effect



Tunnelling effect



Interpretation

The ergodic constant λ_{HJ} is the analog of the Perron eigenvalue in the case of optimal control.

We have in general

$$\sup_{\alpha} \lambda_P(\alpha) \leqslant \lambda_{HJ}.$$

Interpretation

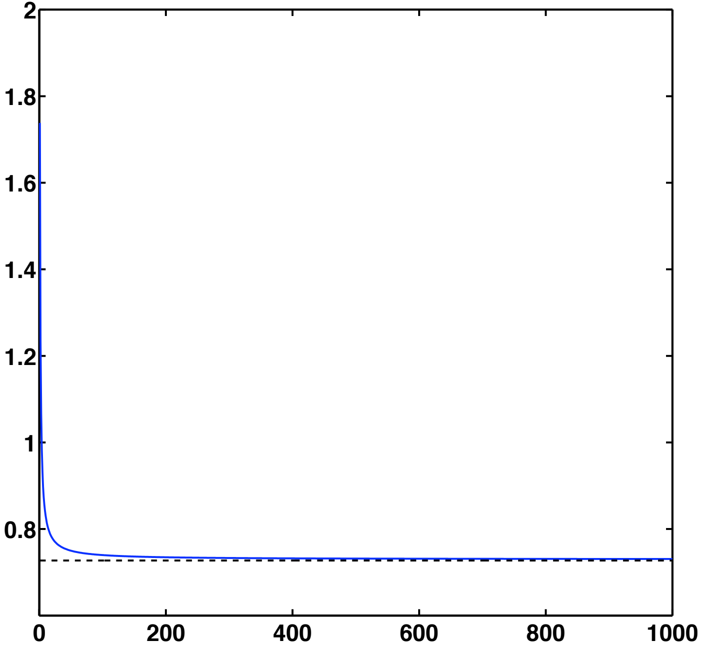
The ergodic constant λ_{HJ} is the analog of the Perron eigenvalue in the case of optimal control.

We have in general

$$\sup_{\alpha} \lambda_P(\alpha) \leqslant \lambda_{HJ}.$$

In our case, we observe numerically the identity

$$\sup_{\alpha} \lambda_P(\alpha) = \lambda_{HJ}.$$


2
1.8
1.6
1.4
1.2
1
0.8
0
200
400
600
800
1000