

Nonlinear Integrate & Fire Neuron Models

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- Joint work with: J. Carrillo (UAB), M. Gualdani (UT Austin), M. Schonbek (UCSC)

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- Previous work: M. Cáceres, J. Carrillo, B. Perthame (2011)

The model

Unknown: V = membrane potential

Basic LIF model

$$C_m \frac{dV}{dt} = -g_L(V - V_L) + I(t)$$

- $\tau_m = C_m/g_L \approx 10ms$
 $V_L \approx -70mV$.
- $I(t)$: the external input current.

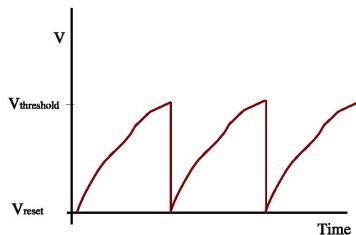
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 $V_L \approx -70mV$.
- $I(t)$: the external input current.
- Firing voltage: threshold value
 $V_F \approx -50mV$.
- Reset voltage: discharged value
 $V_R \approx -60mV$.

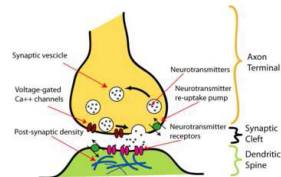


Stochastic Synapse Model

$$I(t) = J_E \sum_{i=1}^{C_E} \sum_j \delta(t - t_{Ej}^i) - J_I \sum_{i=1}^{C_I} \sum_j \delta(t - t_{Ij}^i)$$

Parameters

- Two different Neuron-types: **Inhibitory** (I) or **Excitatory** (E).
Strength of the Synapse: J .
Number of presynaptic neurons: C .
- Spiking times: t_j^i = time of the j^{th} -spike coming from the i^{th} -presynaptic neuron.
- Stochastic Assumption: Neurons spike according to a Poisson process with constant probability of emitting a spike per unit time ν .



Diffusion Approximation

Mean and Variance of $I(t)$:

$$\mu_C = b\nu \quad \text{with} \quad b = C_E J_E - C_I J_I \quad \text{and} \quad \sigma_C^2 = (C_E J_E^2 + C_I J_I^2)\nu.$$

Diffusion Approximation

Several authors (Brunel, Hakim, Renart, Wang, Mattia, del Giudice) propose to approximate the current by

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- 2 The value of $N(t)$ is then computed as the flux of probability that neurons cross the threshold voltage V_F per unit time.
- 3 Average-excitatory (-inhibitory resp.) if $b > 0$ ($b < 0$ resp.).

PDE model (special case)

- Ito's rule: PDE for the evolution of the **probability density** $p(v, t) \geq 0$ of finding neurons at a voltage $v \in (-\infty, V_F]$ at a time $t \geq 0$.

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- **Initial data:** $p(v, 0) = p_I(v)$.

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 - There exists initial data such that develop blow up at finite time
 - There are cases of non-existence or non-uniqueness for the steady state
- Concentrate, for now, on the inhibitory case ($b < 0$)

- Change of variables: Fokker-Plank to heat equation

$$y = e^t v, \quad \tau = \frac{1}{2}(e^{2t} - 1), \quad p(v, t) = e^t w\left(e^t v, \frac{1}{2}(e^{2t} - 1)\right)$$

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- Change of variables: Get rid of w_y term:
 $u(x, \tau) = w(y, \tau), \quad x = y - b \int_0^\tau M(s)\alpha^{-1}(s) ds.$
- Get a free boundary Stefan problem with source
- Related to a price formation equation
(Lasry-Lions, G.-Gualdani, Chayes-G.-Gualdani-Kim,
Caffarelli-Markowich-Pietschmann)

Relation to Stefan problem

$$(P) \quad \left\{ \begin{array}{l} u_t = u_{xx} + M(t)\delta_{x=s(t)}, \quad x < s(t), t > 0, \\ s(t) = -b \int_0^t M(s)\alpha^{-1}(s) ds, \quad t > 0, \\ M(t) = -\left. \frac{\partial u}{\partial x} \right|_{x=s(t)}, \quad t > 0, \end{array} \right.$$

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- 2 The flux across the free boundary $s(t)$ is the jump of the δ at $s_1(t)$: $M(t) := -u_x(s(t), t) = u_x(s_1(t)^-, t) - u_x(s_1(t)^+, t)$

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- 3 If $b < 0$, the free boundary $s(t)$ is an increasing function of time, and $s(t)$, $s_1(t)$ never cross.

- Green's function: $G(x, t, \xi, \tau) = \frac{1}{[4\pi(t-\tau)]^{1/2}} \exp \left\{ -\frac{|x-\xi|^2}{4(t-\tau)} \right\}$

Integral formulation *a la Friedman*

- Green's function: $G(x, t, \xi, \tau) = \frac{1}{[4\pi(t-\tau)]^{1/2}} \exp \left\{ -\frac{|x-\xi|^2}{4(t-\tau)} \right\}$
- Duhamel's formula:

$$u(x, t) = \underbrace{\int_{-\infty}^0 G(x, t, \xi, 0) u_I(\xi) d\xi}_{\text{homogeneous heat equation}} - \underbrace{\int_0^t M(\tau) G(x, t, s(\tau), \tau) d\tau}_{\text{Takes care of free boundary}} + \underbrace{\int_0^t M(\tau) G(x, t, s_1(\tau), \tau) d\tau}_{\text{Delta function at } s_1(t)}$$

Integral formula for M

$$\begin{aligned}M(t) &= \partial_x u(s(t), t) \\&= -2 \int_{-\infty}^0 G(s(t), t, \xi, 0) u'_l(\xi) d\xi \\&\quad + 2 \int_0^t M(\tau) G_x(s(t), t, s(\tau), \tau) d\tau \\&\quad - 2 \int_0^t M(\tau) G_x(s(t), t, s_1(\tau), \tau) d\tau \\&=: \Theta(M)(t)\end{aligned}$$

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Fixed point argument: $\Theta : C_{\sigma, m} \rightarrow C_{\sigma, m}$, where

$$C_{\sigma, m} := \{M \in \mathcal{C}([0, \sigma]) : \|M\| \leq m\}$$

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Contraction for σ small, where $\sigma = \sigma(\|u'_l\|, v_R)$

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Corollary - short time existence for u

$\exists!$ “regular” solution u for $t \in (0, T)$

Global existence ($b < 0$)

Lemma - no break up time

$\exists \mu > 0$ small enough such that, for any $t_0 > 0$, if

$$\sup_{x \in (-\infty, s(t_0 - \mu)]} |u_x(x, t_0 - \mu)| < \infty,$$

then also

$$\sup_{t_0 - \mu < t < t_0} M(t) < \infty.$$

Here μ is independent of t_0 .

Proof: more careful estimates in the integral formula

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Theorem ($b < 0$)

There exists a unique regular solution u for all $t > 0$.

Blow up when $b > 0$

Criterium

Solution exists up to time T^* , where

$$T^* = \sup\{t < 0 : M(t) < \infty\}.$$

Proof:

M bounded $\Rightarrow u_x$ bounded $\Rightarrow$ Can extend solution past T^*

Asymptotics for $b = 0$ (linear case - no free boundary)

$$(P_0) \quad \begin{cases} p_t = \partial_v [vp] + p_{vv} + N(t)\delta_{v=v_R}, := \mathcal{L}p, & v < 0, t > 0, \\ N(t) = - \left. \frac{\partial p}{\partial v} \right|_{v=0}, & t > 0, \\ p(-\infty, t) = 0, \quad p(0, t) = 0, & t > 0, \\ p(v, 0) = p_I(v), & v < 0. \end{cases}$$

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Known (Caceres-Carrillo-Perthame):

- There is a unique equilibrium.
- Solution decays exponentially fast to equilibrium.
- Entropy methods + Poincaré inequality
- Weight does not satisfy Muckenhoupt condition

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Question: decay.... at what rate **????**

There exists a one parameter family of equilibria

$$p_{\infty}(v) = \begin{cases} e^{-v^2/2} & v \in (-\infty, V_R), \\ \alpha_0 e^{-v^2/2} \int_v^0 e^{v^2/2} dv & v \in (V_R, 0], \end{cases}$$

for

$$\alpha_0 := \left(\int_{V_R}^0 e^{v^2/2} dv \right)^{-1}.$$

Standard Fokker-Planck equation

- Compute the eigenvalues: $\mathcal{L}p = \lambda p$, where

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- Norm:

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- Eigenvalues are

$$\lambda = -n, \quad n \in \mathbb{N}.$$

- Eigenfunctions are given by the Hermite polynomials H_n as

$$p_n(v) = H_n(v) e^{-v^2/2}.$$

Back to our model

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Theorem

- $\lambda = 0$ is an eigenvalue for $\mathcal{L}$ with one-dimensional eigenspace that corresponds to the steady state.

$$(P_0) \quad \begin{cases} p_t = \partial_v [vp] + p_{vv} + N(t)\delta_{v=V_R}, := \mathcal{L}p, & v < 0, t > 0, \\ N(t) = - \left. \frac{\partial p}{\partial v} \right|_{v=0}, & t > 0, \\ p(-\infty, t) = 0, \quad p(0, t) = 0, & t > 0, \\ p(v, 0) = p_I(v), & v < 0. \end{cases}$$

Theorem

- $\lambda = 0$ is an eigenvalue for $\mathcal{L}$ with one-dimensional eigenspace that corresponds to the steady state.
- No other $\lambda \in \mathbb{C}$ can be an eigenvalue for $\mathcal{L}$, except for a countable set of values for V_R .

Back to our model

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Conclusion: the spectral gap may have nothing to do with the exponential decay to equilibrium !!!

Thank you!!