Stochastic Schrödinger equations with unbounded coefficients [1]

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Outline

Example 1

Paul trap - Fluctuations in the location of the center

State space: $\mathfrak{h} = L^2(\mathbb{R}, \mathbb{C})$

Hamiltonian: $H = -\frac{1}{2M}\frac{d^2}{dx^2} + \frac{1}{2}M\omega^2 x^2$, with $M > 0$ and $\omega \in \mathbb{R}$

$L_1 = -i\eta x$, with $\eta > 0$.

$$X_t = X_0 + \int_0^t \left(-iH + \frac{1}{2} L_1^* L_1 \right) X_s ds + \int_0^t L_1 X_s dW_s^k$$

M.E. Ghem, K.M. O'Hara, T.A. Savard, and J.E. Thomas - Phys. Rev. A (1998)
S. Schneider and G. J. Milburn - Phys. Rev. A (1999)
T. Grotz, L. Heaney, and W. Strunz - Phys. Rev. A (2006)

Example 2

Application of intense laser pulse to the hydrogen-like atom

State space: $\mathfrak{h} = L^2(\mathbb{R}, \mathbb{C})$

Hamiltonian: $H = -\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{\sqrt{x^2+a^2}} + xF(t)$, with

$$F(t) = F_0 \sin(\beta t + \delta) \cdot \begin{cases} \sin(\pi t/(2\tau)), & \text{if } t < \tau \\ 1, & \text{if } \tau \le t \le T - \tau \\ \cos^2(\pi(t + \tau - T)/(2\tau)), & \text{if } T - \tau \le t \le T \end{cases} .$$

$L_1 = -i\eta x$

$\beta, \eta, \delta \in \mathbb{R}$ and $a, F_0, \tau, T > 0$

$$X_t = X_0 + \int_0^t \left(-iH + \frac{1}{2}L_1^* L_1\right) X_s ds + \int_0^t L_1 X_s dW_s^k$$

K.P. Singh and J.M. Rost - Phys. Rev. A (2007)

Example 3 (Quantum measurement)

Simultaneous measurement of position and momentum

State space: $\mathfrak{h} = L^2(\mathbb{R}, \mathbb{C})$
Hamiltonian: $H = -\alpha \frac{d^2}{x^2} + \beta x^2$, with $\alpha \geq 0$ and $\beta \in \mathbb{R}$
$L_1 = \frac{\kappa}{\sigma} x$ and $L_2 = -i\kappa\sigma \frac{d}{dx}$, with $\kappa, \sigma \in]0, \infty[$

$$Y_t = Y_0 + \int_0^t G(Y_s)\, ds + \sum_{k=1}^{2} \int_0^t L_k(Y_s)\, dB_s^k$$

- $G(y) = \left(-iH - \frac{1}{2}\sum_{k=1}^{2} L_k^* L_k\right) y + \sum_{k=1}^{2} \left(\mathrm{Re}\,\langle y, L_k y\rangle L_k y - \frac{1}{2}\mathrm{Re}^2\,\langle y, L_k y\rangle\, y\right)$
- $L_k(y) = L_k y - \mathrm{Re}\,\langle y, L_k y\rangle\, y$

A.J. Scott and G.J. Milburn - Phys. Rev. A (2001)
J. Gough and A. Sobolev - Phys. Rev. A (2004)
A. Bassi and D. Dürr - Europhys. Lett. (2008)

Example 4

Quantum oscillator

State space: $\mathfrak{h} = l^2(\mathbb{Z}_+)$

$H = i\beta_1 \left(a^\dagger - a\right) + \beta_2 N + \beta_3 \left(a^\dagger\right)^2 a^2$

$L_1 = \alpha_1 a$, $L_2 = \alpha_2 a^\dagger$, $L_3 = \alpha_3 N$,

$L_4 = \alpha_4 a^2$, $L_5 = \alpha_5 \left(a^\dagger\right)^2$, $L_6 = \alpha_6 N^2$

$(e_n)_{n\in\mathbb{Z}_+}$: orthonormal basis of $l^2(\mathbb{Z}_+)$

$$a^\dagger e_n = \sqrt{n+1}\, e_{n+1}, \qquad a e_n = \begin{cases} 0, & \text{if } \quad n = 0 \\ \sqrt{n} e_{n-1}, & \text{if } \quad n > 0 \end{cases}, \quad N = a^\dagger a$$

$$X_t = X_0 + \int_0^t \left(-iH + \frac{1}{2}\sum_{k=1}^{6} L_1^* L_1\right) X_s ds + \sum_{k=1}^{6} \int_0^t L_1 X_s dW_s^k$$

Linear stochastic Schrödinger equation

Belavkin (Physics Letter A, 1989)

$$X_t(x) = x + \int_0^t G(s) X_s(x)\, ds + \sum_{k=1}^{\infty} \int_0^t L_k(s) X_s(x)\, dW_s^k \quad (1)$$

$$G(s) = -iH(s) - \frac{1}{2} \sum_{k=1}^{\infty} L_k(s)^* L_k(s)$$

Non-linear stochastic Schrödinger equation

Non-linear stochastic evolution equation

$$Y_t^y = y + \int_0^t G\left(s, Y_s^y\right) ds + \sum_{k=1}^{\infty} \int_0^t L_k\left(s, Y_s^y\right) dB_s^k \qquad (2)$$

- $\|y\| = 1$
- $B^1, B^2, \ldots$: independent Brownian motions.
- $L_k(s,x) = L_k(s)\,x - \operatorname{Re}\langle x, L_k(s)\,x\rangle\, x.$
- $G(s,x) = G(s)\,x + \sum_{k=1}^{\infty}\left(\operatorname{Re}\langle x, L_k(s)\,x\rangle\, L_k(s)\,x - \frac{1}{2}\operatorname{Re}^2\langle x, L_k(s)\,x\rangle\, x\right).$

Hypothesis 1

Hypothesis 1

Let C be a self-adjoint positive operator in $\mathfrak{h}$ such that:

- For any $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$ and $T > 0$, the linear stochastic Schrödinger equation (1) has a unique strong C-solution on $[0, T]$ with initial datum ξ.
- For all $x \in \mathcal{D}(C)$ and $t \geq 0$,

$$2\Re \langle x, Gx\rangle + \sum_{k=1}^{\infty} \|L_k x\|^2 = 0.$$

- For any $x \in \mathcal{D}(C)$ and $t \geq 0$, $\|G(t)x\|^2 \leq K(t)\|x\|_C^2$.

$X_t(\xi)$ is strong C-solution iff

- $\mathbb{E}\|X_t(\xi)\|^2 \leq \mathbb{E}\|\xi\|^2$, $X_t(\xi) \in \mathcal{D}(C)$ a.s. and $\sup_{s\in[0,t]} \mathbb{E}\|CX_s(\xi)\|^2 < \infty$.
- $X_t(\xi) = \xi + \int_0^t G(s)\pi_C(X_s(\xi))\,ds + \sum_{\ell=1}^{\infty} \int_0^t L_\ell(s)\pi_C(X_s(\xi))\,dW_s^\ell$ $\mathbb{P}$-a.s.

Existence and uniqueness

Theorem (C.M. M and R. Rebolledo (Ann. Appl. Probab. (2008)) - F. Fagnola and C.M. M. (2011)

Let C satisfy Hypothesis 1.
Suppose that θ is a probability measure on $\mathfrak{B}(\mathfrak{h})$ such that $\theta(\mathcal{D}(C) \cap \{y \in \mathfrak{h} : \|y\| = 1\}) = 1$ and $\int_{\mathfrak{h}} \|Cy\|^2 \theta(dx) < \infty$.

Then the non-linear stochastic Schrödinger equation (2) has a unique C-solution $\left(\mathbb{Q}, (Y_t)_{t\geq 0}, (B_t)_{t\geq 0}\right)$ with initial law θ.

Evolution of density operators

Quantum master equation

$$\frac{d}{dt}\rho_t(\varrho) = \rho_t(\varrho)\,G(t)^* + G(t)\,\rho_t(\varrho) + \sum_{k=1}^{\infty} L_k(t)\,\rho_t(\varrho)\,L_k(t)^* \qquad (3)$$

$$\rho_0(\varrho) = \varrho$$

Previous results

- Existence of minimal solution: Davies (Rep. Math. Phys., 1977)
- Uniqueness: Chebotarev, Fagnola (J. Funct. Anal., 1998)
- Regularity of solutions: Davies (Comm. math. phys., 1977, Neutron diffusion equation); Chebotarev, García and Quezada (Publ. Res. Inst. Math. Sci. Kokyuroku, 1998, General results); Arnold and Sparber (Commun. Math. Phys., 2004,Diffusion models with Hartree interaction)

- Is $tr\left(A\rho_t(\varrho)\right)$ well-posed?

Regular density operator

Definition

Let C be an self-adjoint positive operator.
Then $\varrho \in \mathfrak{L}^{+}_{1,C}(\mathfrak{h})$ if and only if

- ϱ is a positive trace class operator in $\mathfrak{h}$.
- There is an orthonormal basis $(u_n)_{n\in\mathbb{Z}_+}$ of $\mathfrak{h}$ and a sequence of non-negative real numbers $(\lambda_n)_{n\in\mathbb{Z}_+}$ such that:
  - $\varrho = \sum_{n\in\mathbb{Z}_+} \lambda_n |u_n\rangle\langle u_n|$.
  - $\sum_{n\in\mathbb{Z}_+} \lambda_n \|Cu_n\|^2 < +\infty$

Characterizations of $\mathfrak{L}^{+}_{1,C}(\mathfrak{h})$: Chebotarev, Garcia and Quezada (1998)

Lemma

ϱ is C-regular if and only if there exists $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$ for which $\varrho = \mathbb{E}\,|\xi\rangle\langle\xi|$.

Evolution of density operators

Theorem

Suppose that Hypothesis 1 holds. Then, for every $t \geq 0$ there exists a unique operator ρ_t belonging to $\mathfrak{L}\left(\mathfrak{L}_1\left(\mathfrak{h}\right)\right)$ such that for each C-regular operator ϱ we have

$$\rho_t\left(\varrho\right) = \mathbb{E}\left|Y_t^{\xi}\right\rangle\left\langle Y_t^{\xi}\right| = \mathbb{E}\left|X_t^{\xi}\right\rangle\left\langle X_t^{\xi}\right|,$$

whenever ξ is an arbitrary random variable satisfying $\xi \in L_C^2\left(\mathbb{P},\mathfrak{h}\right)$ and $\varrho = \mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$.

Theorem

Under Hypothesis 1,
$\rho_t\left(\mathfrak{L}_1^+\left(\mathfrak{h}\right)\right) \subset \mathfrak{L}_1^+\left(\mathfrak{h}\right)$ and $\rho_t\left(\mathfrak{L}_{1,C}^+\left(\mathfrak{h}\right)\right) \subset \mathfrak{L}_{1,C}^+\left(\mathfrak{h}\right)$.

Mean values

Lemma

Suppose that:

- $\varrho = \mathbb{E}\,|\xi\rangle\langle\xi|$ for $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$.
- $A \in \mathfrak{L}\left(\left(Dom\left(C\right), \langle\cdot,\cdot\rangle_C\right), \mathfrak{h}\right)$.

Then $A\varrho = \mathbb{E}\,|A\xi\rangle\langle\xi|$ and $tr\left(A\varrho\right) = \mathbb{E}\,\langle\xi, A\xi\rangle$.

If in addition $Dom\left(C\right) \subset Dom\left(A^*\right)$, we have

- $\varrho A = \mathbb{E}\,|\xi\rangle\langle A^*\xi|$.
- $tr\left(\varrho A\right) = \mathbb{E}\,\langle\xi, A\xi\rangle$.

Existence and uniqueness of solutions for QMEs

Theorem (C.M. Ann. Probab. (to appear))

Consider the autonomous case. Let Hypothesis 1 hold. Then for any $A \in \mathfrak{L}(\mathfrak{h})$ and $t \geq 0$,

$$\frac{d}{dt} tr\left(A\rho_t(\varrho)\right) = tr\left(A\left(G\rho_t(\varrho) + \rho_t(\varrho)G^* + \sum_{k=1}^{\infty} L_k \rho_t(\varrho) L_k^*\right)\right). \tag{4}$$

Moreover, $(\rho_t)_{t\geq 0}$ is the unique semigroup of bounded operators on $\mathfrak{L}_1(\mathfrak{h})$ such that:

i) $\sup_{t\in[0,T]} \left\|\rho_t\right\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))} < \infty$.

ii) For each $x \in Dom(C)$, the function $t \mapsto tr\left(\rho_t(|x\rangle\langle x|)A\right)$ is continuous provided $A \in \mathfrak{L}(\mathfrak{h})$.

iii) Relation (4) holds with $\varrho = |x\rangle\langle x|$ whenever $x \in Dom(C)$.

Hypothesis 2 (non-explosion condition)

Let C be a self-adjoint positive operator in $\mathfrak{h}$ with the properties:

- For any $x \in \mathcal{D}(C)$, $\|G(t)x\|^2 \leq K(t)\|x\|_C^2$.
- For all $x \in \mathcal{D}(C)$, $\|L_k(t)x\|^2 \leq K(t)\|x\|_C^2$.
- There exists $\alpha \geq 0$ and a core $\mathfrak{D}_1$ of C^2 such that for all $x \in \mathfrak{D}_1$,

$$2\Re\left\langle C^2 x, G(t)x\right\rangle + \sum_{k=1}^{\infty}\|CL_k(t)x\|^2 \leq \alpha(t)\|x\|_C^2 .$$

- There exist a core $\mathfrak{D}_2$ of C such that for any x in $\mathfrak{D}_2$,

$$2\Re\langle x, G(t)x\rangle + \sum_{k=1}^{\infty}\|L_k(t)x\|^2 \leq 0.$$

Theorem (C.M. M. , F. Fagnola, 2011)

Assume that Hypothesis 2 holds.
Let $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$.
Then, the linear stochastic Schrödinger equation (1) has a unique strong C-solution $(X_t(\xi))_{t\geq 0}$ with initial datum ξ.
Moreover,

$$\mathbb{E}\left\|CX_t(\xi)\right\|^2 \leq \exp(\alpha t)\left(\mathbb{E}\left\|C\xi\right\|^2 + \alpha t \mathbb{E}\left\|\xi\right\|^2 + \beta t\right).$$

Model 1

Consider $\mathfrak{h} = L^2\left(\mathbb{R}^d, \mathbb{C}\right)$. Let the Hamiltonian be

$$H(t) = -\alpha\Delta + i\sum_{j=1}^{d}\left(A^j(t,\cdot)\partial_j + \partial_j A^j(t,\cdot)\right) + V(t,\cdot),$$

where $t \geq 0$, $\alpha \geq 0$, and $V, A^1, \ldots, A^d$ are real-valued measurable smooth functions on $[0, +\infty[\times\mathbb{R}^d$.

For a given $m \in \mathbb{N}$ and for all $t \geq 0$ choose

$$L_\ell(t) = \begin{cases} \sum_{k=1}^{d}\sigma_{\ell k}(t,\cdot)\,\partial_k + \eta_\ell(t,\cdot), & \text{if } 1 \leq \ell \leq m \\ 0, & \text{if } \ell > m \end{cases},$$

where $\sigma_{\ell k}$, η_ℓ : $[0, +\infty[\times\mathbb{R}^d \to \mathbb{C}$ are complex-valued measurable smooth functions.

Hypothesis 3

Adopt Model 1. Define $G(t) = -iH(t) - \frac{1}{2}\sum_{\ell=1}^{m} L_\ell^*(t) L_\ell(t)$.

(H3.1)

Suppose that: $V(t,\cdot) \in C^2(\mathbb{R}^d, \mathbb{R})$, $A^j(t,\cdot) \in C^3(\mathbb{R}^d, \mathbb{R})$,

$\max\{|V(t,x)|, |\Delta V(t,x)|, |\partial_j(\Delta A^j)|\} \le K(t)\left(1 + |x|^2\right)$,

$\max\{|\partial_j V(t,x)|, |A^j(t,x)|, |(\partial_{j'}\partial_j A^j)(t,x)|\} \le K(t)(1 + |x|)$

$|\partial_{j'} A^j(t,x)| \le K(t)$

$$H(t) = -\alpha\Delta + i\sum_{j=1}^{d}\left(A^j(t,\cdot)\partial_j + \partial_j A^j(t,\cdot)\right) + V(t,\cdot)$$

Hypothesis 3

(H3.2)

$|\sigma_{\ell k}(t,\cdot)| \leq K(t)$,
$\eta_\ell(t,\cdot) \in C^3(\mathbb{R}^d, \mathbb{C})$ and the absolute values of all the partial derivatives of $\eta_\ell(t,\cdot)$ from the first up to the third order are bounded by $K(t)$.

At least one of the following conditions holds:

- $|\eta_\ell(t,\cdot)| \leq K(t)$, $\sigma_{\ell k}(t,\cdot) \in C^3(\mathbb{R}^d, \mathbb{C})$, and the absolute values of all partial derivatives of $\sigma_{\ell k}(t,\cdot)$ up to the third order are dominated by $K(t)$.
- $(t,x) \mapsto \sigma_{\ell k}(t,x)$ does not depend on x and $|\eta_\ell(t,0)| \leq K(t)$.

$$L_\ell(t) = \sum_{k=1}^{d} \sigma_{\ell k}(t,\cdot)\,\partial_k + \eta_\ell(t,\cdot)$$

Theorem (CMM, F. Fagnola (2011))

Suppose that Hypothesis 3 holds.
Set $C = -\Delta + |x|^2$.
Let ξ be a $\mathfrak{F}_0$-measurable random variable taking values in $L^2(\mathbb{R}^d, \mathbb{C})$ such that $\mathbb{E}\|\xi\|^2 = 1$ and $\mathbb{E}\|C\xi\|^2 < \infty$.

Then the linear SSE (1) has a unique strong C-solution with initial datum ξ. Moreover, $\mathbb{E}\|X_t(\xi)\|^2 = \|\xi\|^2$ for all $t > 0$.

Hypothesis 4

Let C satisfy Hypotheses 1. Suppose that:
For all $t \geq 0$ and any x belonging to a core of C,

$$\sum_{\ell=1}^{\infty} \left\| C^{1/2} L_{\ell}(t) x \right\|^{2} \leq K(t) \|x\|_{C}^{2}.$$

Let $A = B_1^* B_2$, where B_1, B_2 are operators in $\mathfrak{h}$ such that:

- For all $x \in \mathcal{D}(C^{1/2})$, $\max\{\|B_1 x\|^2, \|B_2 x\|^2\} \leq K \|x\|_{C^{1/2}}^2$.
- $\max\left\{\|Ax\|^2, \|A^* x\|^2\right\} \leq K \|x\|_C^2$ whenever $x \in \mathcal{D}(C)$.

Theorem F. Fagnola and C.M. M. (2012)

Let Hypothesis 4 hold.
Assume the existence and uniqueness of a strong C-solution to the linear SSE (1) with initial datum $\xi \in L^2_C(\mathbb{P}; \mathfrak{h})$ on any bounded interval.

Then, for all $t \geq 0$ we have

$$\begin{aligned}\mathbb{E}\left\langle X_t(\xi), AX_t(\xi)\right\rangle &= \mathbb{E}\langle \xi, A\xi\rangle + \int_0^t \mathbb{E}\left\langle A^* X_s(\xi), GX_s(\xi)\right\rangle ds \\ &+ \int_0^t \mathbb{E}\left\langle GX_s(\xi), AX_s(\xi)\right\rangle ds \\ &+ \int_0^t \left(\sum_{\ell=1}^{\infty} \mathbb{E}\left\langle B_1 L_\ell X_s(\xi), B_2 L_\ell X_s(\xi)\right\rangle\right) ds\end{aligned}$$

Theorem F. Fagnola and C.M. M. (2012)

Assume the context of Model 1, together with Hypothesis 3.
For any $j = 1, 2$, let B_j be either $\partial_k \left[a_j\right]$, $\left[b_j\right] \partial_k$ or $\left[c_j\right]$, where $k = 1, \ldots, d$, $a_j \in C^2\left(\mathbb{R}^d, \mathbb{R}\right)$ and $b_j, c_j \in C^1\left(\mathbb{R}^d, \mathbb{R}\right)$.
Suppose that:
$\max\left\{\left|a_j(x)\right|, \left|b_j(x)\right|\right\} \leq K$,
$\max\left\{\left|c_j(x)\right|, \left|\partial_l a_j(x)\right|, \left|\partial_l b_j(x)\right|\right\} \leq K(1 + |x|)$, and
$\max\left\{\left|\partial_l c_j(x)\right|, \left|\partial_k \partial_l a_j(x)\right|\right\} \leq K\left(1 + |x|^2\right)$.

If $A = B_1^* B_2$ and $\xi \in L^2_{-\Delta + |x|^2}(\mathbb{P}; \mathfrak{h})$, then

$$
\begin{aligned}
\mathbb{E}\left\langle X_t(\xi), A X_t(\xi)\right\rangle = {} & \mathbb{E}\langle \xi, A\xi\rangle + \int_0^t \mathbb{E}\left\langle A^* X_s(\xi), G X_s(\xi)\right\rangle ds \\
& + \int_0^t \mathbb{E}\left\langle G X_s(\xi), A X_s(\xi)\right\rangle ds \\
& + \int_0^t \left(\sum_{\ell=1}^{\infty} \mathbb{E}\left\langle B_1 L_\ell X_s(\xi), B_2 L_\ell X_s(\xi)\right\rangle\right) ds
\end{aligned}
$$

Example 1

State space: $\mathfrak{h} = L^2(\mathbb{R}, \mathbb{C})$

Hamiltonian: $H = -\frac{1}{2M}\frac{d^2}{dx^2} + \frac{1}{2}M\omega^2 x^2$, with $M > 0$ and $\omega \in \mathbb{R}$

$L_1 = -i\eta x$, with $\eta > 0$.

Theorem F. Fagnola and C.M. M. (2012)

In Example 1, for all $t \geq 0$ we have

$$\mathbb{E}\left\langle X_t(\xi), HX_t(\xi)\right\rangle = \mathbb{E}\left\langle \xi, H\xi\right\rangle + \frac{1}{2M}\eta^2 t.$$

Hypothesis 5

Hypothesis 5

Suppose that there exist: (i) a self-adjoint positive operator D in $\mathfrak{h}$ satisfying Hypothesis 1; and (ii) a probability measure Γ on $\mathfrak{B}(\mathfrak{h})$ such that:

- For any $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{h})$, $\Gamma(A) = \int_{\mathfrak{h}} P_t(x, A)\, \Gamma(dx)$
- $\Gamma(Dom(D) \cap \{x \in \mathfrak{h} : \|x\| = 1\}) = 1$
- $\int_{\mathfrak{h}} \|Dz\|^2 \, \Gamma(dz) < \infty$

Here $P_t(x, A) = \begin{cases} \mathbb{Q}_x(Y_t^x \in A), & x \in Dom(D) \\ \delta_x(A), & x \notin Dom(D) \end{cases}$

Sufficient condition: AAP (2008) C.M.M - R. Rebolledo

Regular invariant density operator

Theorem

Let D satisfy Hypothesis 5.
Then, there exists a D-regular operator ϱ_∞ such that

$$\rho_t\left(\varrho_\infty\right) = \varrho_\infty$$

for all $t \geq 0$.

Example 4 (quantum oscillator)

State space: $\mathfrak{h} = l^2(\mathbb{Z}_+)$

$(e_n)_{n\in\mathbb{Z}_+}$: orthonormal basis of $l^2(\mathbb{Z}_+)$

$a^\dagger e_n = \sqrt{n+1}e_{n+1}$, $\quad ae_n = \begin{cases} 0, & \text{if} \quad n=0 \\ \sqrt{n}e_{n-1}, & \text{if} \quad n>0 \end{cases}$

$N = a^\dagger a$

$$H = i\beta_1 (a^\dagger - a) + \beta_2 N + \beta_3 (a^\dagger)^2 a^2$$
$$L_1 = \alpha_1 a,\ L_2 = \alpha_2 a^\dagger,\ L_3 = \alpha_3 N,$$
$$L_4 = \alpha_4 a^2,\ L_5 = \alpha_5 (a^\dagger)^2,\ L_6 = \alpha_6 N^2$$

Unbounded observables

- N: number of photons
- $i(a^\dagger - a)/\sqrt{2}$: The position operator
- $(a^\dagger + a)/\sqrt{2}$: The momentum operator

Example 4

$$H = i\beta_1 \left(a^\dagger - a\right) + \beta_2 N + \beta_3 \left(a^\dagger\right)^2 a^2, \; L_1 = \alpha_1 a, \; L_2 = \alpha_2 a^\dagger,$$
$$L_3 = \alpha_3 N, \, L_4 = \alpha_4 a^2, \, L_5 = \alpha_5 \left(a^\dagger\right)^2, \, L_6 = \alpha_6 N^2$$

Theorem

In the set-up of Example 4 we assume

$$|\alpha_4| \geq |\alpha_5| \, .$$

If $p \geq 4$, then there exists a unique N^p-regular solution to the quantum master equation (3).

In addition, there exists a N^p-regular operator ϱ_∞ which is invariant for (3) provided that either
$|\alpha_4| > |\alpha_5|$ or
$|\alpha_4| = |\alpha_5|$ with $|\alpha_2|^2 - |\alpha_1|^2 + 4\,(2p+1)\,|\alpha_4|^2 < 0$.

Conclusion

- We can study theoretical properties of the quantum Markovian master equations with the help of stochastic Schrödinger equations.

- We obtain an Ehrenfest's-type theorem for open quantum systems.

- We can prove rigorously the heating of ion traps in a simple model.

- In many physical situations, there exists a regular stationary solution for the quantum master equations.

Thank you very much!