The future of some Bianchi A spacetimes with an ensemble of free falling particles

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Overview

- Einstein equations and motivation
- Previous important results
- What is a Bianchi spacetime?
- What is the Vlasov equation?
- Results and an example of a bootstrap argument
- Outlook

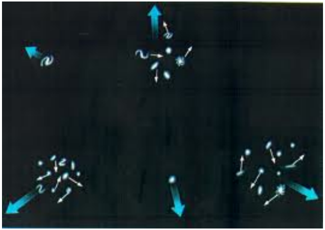
The Einstein equations

- A spacetime is $(M, g_{\alpha\beta})$; signature $-+++$
- In mathematical cosmology one usually assumes $M = I \times G$ where G is spatially compact
- Einstein equations (with c=G=1):

$$G_{\alpha\beta} + g_{\alpha\beta}\Lambda = 8\pi T_{\alpha\beta}$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is divergence-free

The Universe as a fluid



The equation of state $P = f(\rho)$ relates the pressure P with the energy density ρ. The velocity of the fluid/observer is u^α

$$T_{\alpha\beta} = (\rho + P)u_\alpha u_\beta + Pg_{\alpha\beta}$$

The Euler equations of motion coincide with the requirement that $T_{\alpha\beta}$ has to be divergence-free. In the Matter-dominated Era $P = 0$ which corresponds to **dust**

Late-time behaviour of the Universe with a cosmological constant Λ:

- The Cosmic no hair conjecture
- $\exists\Lambda => $ **Vacuum** $+\Lambda$ **at late times** (Gibbons-Hawking 1977, Hawking-Moss 1982)
- Non Bianchi IX homogeneous models with a perfect fluid (Wald 1983)
- Non-linear Stability of 'Vacuum $+\Lambda$' (Friedrich 1986)
- Non-linear Stability of FLRW for $1 < \gamma < \frac{4}{3}$-fluid (Rodnianski-Speck 2011)
- *For Bianchi except IX and Vlasov (Lee 2004)*

What about the situation $\Lambda = 0$?

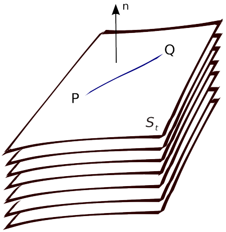
- Mathematically more difficult, since no exponential behaviour
- Late-time asymptotics are well understood for perfect fluid
- Stability of the matter model?
- Stability of the perfect fluid model at late times:
- **Is the Einstein-Vlasov system well-approximated by the Einstein-dust system for an expanding Universe?**

Why Vlasov?

- Vlasov = Boltzmann without collision term
- Nice mathematical properties
- More 'degrees of freedom'
- Kinetic description $f(t, x^a, p^a)$ is often used in (astro)physics
- A starting point for the study of non-equilibrium
- Galaxies when they collide they do not collide
- Plasma is well aproximated by Vlasov

What is a Bianchi spacetime?

- A spacetime is said to be (spatially) *homogeneous* if there exist a one-parameter family of spacelike hypersurfaces S_t foliating the spacetime such that for each t and for any points $P, Q \in S_t$ there exists an isometry of the spacetime metric ${}^4g$ which takes P into Q
- It is defined to be a *spatially homogeneous* spacetime whose isometry group possesses a 3-dim subgroup G that acts *simply transitively* on the spacelike orbits (manifold structure is $M = I \times G$).

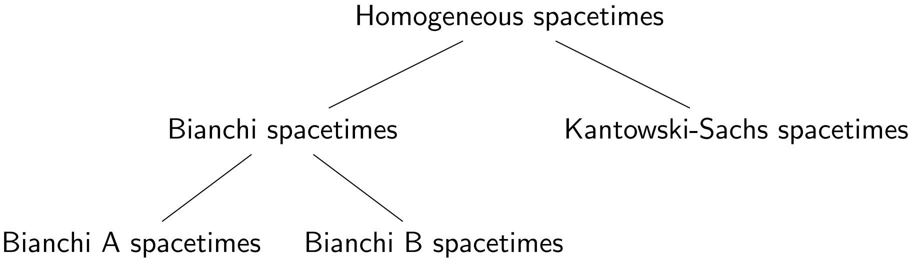


- Bianchi spacetimes have 3 Killing vectors and they can be classified by the structure constants C^i_{jk} of the associated Lie algebra
- $[\xi_j, \xi_k] = C^i_{jk}\xi_i$
- They fall into 2 catagories: A and B
- Bianchi class A is equivalent to $C^i_{ji} = 0$ (unimodular)
- In this case $\exists$ unique symmetric matrix with components ν^{ij} such that $C^i_{jk} = \epsilon_{jkl}\nu^{li}$
- Relation to Geometrization of 3-manifolfds (Reiris 2005)

Classification of Bianchi types class A

Type	ν_1	ν_2	ν_3
I	0	0	0
II	1	0	0
VI_0	0	1	-1
VII_0	0	1	1
VIII	-1	1	1
IX	1	1	1

Subclasses of homogeneous spacetimes



- FLRW closed $\subset$ Bianchi IX
- FLRW flat $\subset$ Bianchi I and Bianchi VII_0
- FLRW open $\subset$ Bianchi V and VII_h with $h \neq 0$

Collisionless matter

- Vlasov equation: $L(f) = 0$, f satisfies $p_\alpha p^\alpha = -m^2$

$$L = \frac{dx^\alpha}{ds}\frac{\partial}{\partial x^\alpha} + \frac{dp^a}{ds}\frac{\partial}{\partial p^a}$$

- Geodesic equations

$$\frac{dx^\alpha}{ds} = p^\alpha$$

$$\frac{dp^a}{ds} = -\Gamma^a_{\beta\gamma} p^\beta p^\gamma$$

- Geodesic spray

$$L = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^a_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^a}$$

Connection to the Einstein equation

- Energy-momentum tensor

$$T_{\alpha\beta} = \int f(x^\alpha, p^a) p_\alpha p_\beta \mu$$

with

$$\mu = |p_0|^{-1} |\det g|^{\frac{1}{2}} dp^1 dp^2 dp^3$$

Here $\det g$ is the determinant of the spacetime metric. Let us call the spatial part S_{ij} and $S = g^{ij} S_{ij}$

- f is C^1 and of compact support

Vlasov equation with Bianchi symmetry

- Vlasov equation with Bianchi symmetry (in a left-invariant frame where $f = f(t, p_a)$)

$$\frac{\partial f}{\partial t} + (p^0)^{-1} C^d_{ba} p^b p_d \frac{\partial f}{\partial p_a} = 0$$

- From the Vlasov equation it is also possible to define the characteristic curve V_a:

$$\frac{dV_a}{dt} = (V^0)^{-1} C^d_{ba} V^b V_d$$

for each $V_i(\bar{t}) = \bar{v}_i$ given $\bar{t}$.

Time origin choice

- initial data are $(g_{ij}(t_0), k_{ij}(t_0), f(t_0))$ on a 3-dim manifold $S(t_0)$
- Assume that $k < 0$.
- Tecnical choice: without loss of generality $t_0 = -2/k(t_0)$.

"New" variables

$$k_{ab} = \sigma_{ab} - H g_{ab}$$

Hubble parameter ('Expansion velocity')

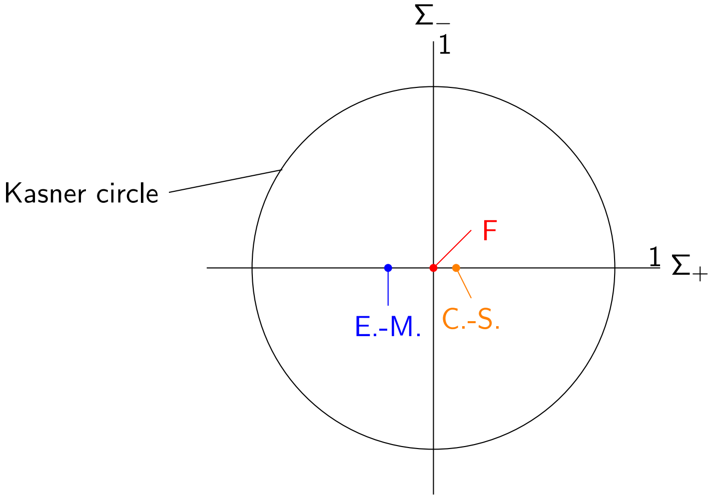
$$H = -\frac{1}{3} k$$

Shear variables ('Anisotropy')

$$\Sigma_+ = -\frac{\sigma_2^2 + \sigma_3^3}{2H}$$

$$\Sigma_- = -\frac{\sigma_2^2 - \sigma_3^3}{2\sqrt{3}H}$$

$$F = \frac{1}{4H^2} \sigma_{ab} \sigma^{ab}$$



The different solutions projected to the $\Sigma_+\Sigma_-$-plane

Results I

- Previous results: Reflection Symmetric Bianchi I; Rendall (1996)
- Reflection symmetry

$$f(p_1, p_2, p_3) = f(-p_1, -p_2, p_3) = f(p_1, -p_2, -p_3)$$

(Implies diagonal metric and $T_{0i} = 0$)

- 'New' result: Drop RS for Bianchi I and small data assumption

Small data assumptions for Bianchi I, II and VI_0

- Close to Einstein-De Sitter/Collins-Stewart/Ellis-Maccallum

$$g_{ED} = t^{\frac{4}{3}}\,\mathrm{diag}(1,1,1)$$
$$g_{CS} = (2t)^{\frac{3}{2}}\,\mathrm{diag}((2t)^{-\frac{1}{2}},1,1)$$
$$g_{EM} = t^{2}\,\mathrm{diag}(1,t^{-1},t^{-1})$$

- Dispersion of Velocities P is bounded, i.e. the spacetime is close to dust, where P is

$$P(t) = \sup\{|p| = (g^{ab}p_a p_b)^{\frac{1}{2}} | f(t,p) \neq 0\}$$

Keys o the proof

- The expected estimates are obtained from the **linearization of the Einstein-dust system** + a corresponding **plausible decay of the velocity dispersion**
- **Bootstrap argument**

Central equations for Bianchi I

$$\partial_t(H^{-1}) = \frac{3}{2} + F + \frac{4\pi S}{3H^2}$$

$$\dot{F} = -3H[F(1 - \frac{2}{3}F - \frac{8\pi S}{9H^2}) - \frac{4\pi}{3H^3}S_{ab}\sigma^{ab}]$$

$$\frac{dV_a}{dt} = 0$$

$$F = \frac{3}{2}(1 - 8\pi T_{00}/3H^2)$$

Expected Estimates

Linearization of the equations corresponding to Einstein-de Sitter with dust

$$F = O(t^{-2})$$
$$P = O(t^{-\frac{2}{3}})$$

Bootstrap assumption

A little worse decay rates then we the ones expect for the interval $[t_0, t_1)$

$$F = A_l(1+t)^{-\frac{3}{2}}$$
$$P = A_m(1+t)^{-\frac{7}{12}}$$

Remark:

$$\frac{S}{H^2} \leq CP^2$$

Estimate of H

$$\partial_t(H^{-1}) = \frac{3}{2} + F + \frac{4\pi S}{3H^2}$$

Integrating and since $t_0 = \frac{2}{3}H^{-1}(t_0)$:

$$H(t) = \frac{1}{\frac{3}{2}t + I} = \frac{2}{3}t^{-1}\frac{1}{1+\frac{2}{3}It^{-1}}$$

with

$$I = \int_{t_0}^{t}(F + \frac{4\pi S}{3H^2})(s)ds$$

With Bootstrap assumptions

$$F + \frac{4\pi S}{3H^2} \leq \epsilon(1+t)^{-\frac{7}{6}}$$

where $\epsilon = C(A_l + A_m^2)$. So I is bounded by ϵ

$$H = \frac{2}{3}t^{-1}(1 + O(t^{-1}))$$

Estimate of the metric

- Define

$$\bar{g}^{ab} = t^{\frac{4}{3}} g^{ab}$$

- Introduce for technical reason a small γ

$$\frac{d}{dt}(t^{-\gamma}\bar{g}^{ab}) \leq t^{-\gamma-1}\bar{g}^{ab}[-\gamma + \frac{4}{3} + 2tH(CH^2F^{\frac{1}{2}} - 1)]$$

Estimate of P

- p_a is constant along the geodesics
- $$P(t) \leq A_m t^{-\frac{2}{3}+\zeta}$$

Estimate of F

- Let us have a look at the evolution equation

$$\dot{F} = -3H[F(1 - \frac{2}{3}F - \frac{8\pi S}{9H^2}) - \frac{4\pi}{3H^3}S_{ab}\sigma^{ab}]$$

- By a contradiction argument

Estimates

Theorem

Consider any C^∞ solution of the Einstein-Vlasov system with Bianchi I-symmetry and with C^∞ initial data. Assume that $F(t_0)$ and $P(t_0)$ are sufficiently small. Then at late times the following estimates hold:

$$\begin{aligned}
H(t) &= \frac{2}{3}t^{-1}(1+O(t^{-1})) \\
F(t) &= O(t^{-2}) \\
P(t) &= O(t^{-\frac{2}{3}})
\end{aligned}$$

More Results

- Previous results: LRS case for Bianchi II: Rendall-Tod (1998) , Rendall-Uggla (2000)
- New result: Bianchi II and reflection symmetric VI_0

Reflection symmetric Bianchi II

The evolution equations are

$$\partial_t(H^{-1}) = \frac{3}{2} - \frac{N_1^2}{24} + \frac{3}{2}(\Sigma_+^2 + \Sigma_-^2) + \frac{4\pi S}{3H^2}$$

$$\dot{\Sigma}_+ = H[\frac{1}{3}N_1^2 - (3 + \frac{\dot{H}}{H^2})\Sigma_+ + \frac{4\pi}{3H^2}(S_2^2 + S_3^3 - 2S_1^1)]$$

$$\dot{\Sigma}_- = H[-(3 + \frac{\dot{H}}{H^2})\Sigma_- + \frac{4\pi}{\sqrt{3}H^2}(S_2^2 - S_3^3)]$$

$$\dot{N}_1 = -N_1 H(4\Sigma_+ + 1 + \frac{\dot{H}}{H^2})$$

and the constraint equation:

$$\Sigma_+^2 + \Sigma_-^2 = 1 - \Omega - \frac{1}{12}N_1^2$$

The Vlasov equation

$$\frac{\partial f}{\partial t} + (p^0)^{-1}p_1(p^2\frac{\partial f}{\partial p_3} - p^3\frac{\partial f}{\partial p_2}) = 0$$

Conclusions

- We have extended the possible initial data which gave us certain asymptotics
- Made a few steps towards the understanding of homogeneous spacetimes
- Bianchi spacetimes and the Vlasov equation are interesting
- PDE-techniques are needed to understand cosmology

Outlook

- Other Bianchi types? Inhomogeneous cosmologies?
- Is it possible to remove the small data assumption(s)?
- Direction of the singularity?