The Harris-Meyn-Tweedie ergodic theorem and some applications in PDE

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Convergence to equilibrium of Markov processes

Studying convergence to equilibrium of Markov processes is a central problem in many fields. Some examples we are interested in:

- Linear Boltzmann and related models
- PDEs in mathematical biology
- Integro-differential / Nonlocal PDE (fractional / nonlocal diffusion)
- Kinetic-type linear PDE, typically hypocoercive

We look at this problem from a PDE point of view and often we want to extend our arguments to:

- Nonlinear equations
- Non-conservative equations

Tools

Many ideas have been used to prove convergence to equilibrium of Markov processes:

Entropy methods Useful for several PDE in mathematical physics. *Hypocoercive* equations are an obstacle. **[Toscani & Villani]**, **[Carrillo & Toscani]**, **[Desvillettes & Villani]**, **[Dolbeault]**, **[Dolbeault, Mouhot & Schmeiser]**, **[Cáceres, Cañizo & Mischler 2011]**, **[Balagué, Cañizo & Gabriel 2013]**, **[Gabriel & Salvarani 2014]**

Operator splitting Often gives a non-constructive spectral gap. "Weyl method", classical in kinetic theory / operator theory. See also **[Gualdani, Mischler & Mouhot]**.

Doeblin / Harris / Meyn-Tweedie The method we're discussing today. Only applicable to linear problems.

Some references

- **[Harris 1956]**
- **[Meyn & Tweedie 1992, 1993]**
- **[Hairer & Mattingly 2011]** - Simplified proof using mass transport distances.
- **[Bakry, Cattiaux & Guillin 2008]**, **[Douc, Fort & Guillin 2009]** - Subexponential version
- **[Gabriel 2017]** - Renewal equation
- **[Dumont & Gabriel 2017]** - Integrate-and-fire neuron model
- **[Eberle, Guillin & Zimmer 2016], [Hu & Wang 2017]** - Kinetic Fokker-Planck
- **[Bansaye, Cloez & Gabriel 2017]** - Doeblin theorem for non-conservative cases

Simple statement: Doeblin's theorem

$(S_t)_{t\geq 0}$ Markov semigroup, where S_t represents the evolution of the law of the process (so S_t acts on measures). Assume:

Doeblin's condition

There exists $t_0 > 0$, $0 < \alpha < 1$ and $\nu \in \mathcal{P}$ with

$$S_{t_0}\mu \geq \alpha\nu \qquad \text{for all } \mu \in \mathcal{P}. \tag{1}$$

Then there exists a unique equilibrium $\mu_* \in \mathcal{P}$ and

$$\|S_t(\mu - \mu_*)\| \leq \frac{1}{1-\alpha} e^{-\lambda t} \|n - n_*\|. \qquad \text{for } t \geq 0 \tag{2}$$

for all $\mu \in \mathcal{P}$, where

$$\lambda := -\frac{\log(1-\alpha)}{t_0} > 0.$$

Doeblin's theorem: proof

The result follows easily from this: first, **if** $\mu_1, \mu_2 \in \mathcal{P}$ **have disjoint support**, by the triangle inequality we have

$$\|S\mu_1 - S\mu_2\|_{\mathrm{TV}} \leq \|S\mu_1 - \alpha\nu\|_{\mathrm{TV}} + \|S\mu_2 - \alpha\nu\|_{\mathrm{TV}}.$$

Now, since $S\mu_i \geq \alpha\nu$, we can write

$$\|S\mu_i - \alpha\nu\|_{\mathrm{TV}} = \int (S\mu_i - \alpha\nu) = \int \mu_1 - \alpha = 1 - \alpha,$$

which gives

$$\|S\mu_1 - S\mu_2\| \leq 2(1-\alpha) = (1-\alpha)\|\mu_1 - \mu_2\|.$$

To get this for any $\mu_1, \mu_2 \in \mathcal{P}$, write

$$\mu_1 - \mu_2 = (\mu_1 - \mu_2)_+ - (\mu_1 - \mu_2)_-$$

and apply previous case.

Harris' theorem

Lyapunov condition

There exists $t_0 > 0$, $0 < \gamma < 1$, $K \geq 0$ and a function $V \colon \Omega \to [1, +\infty)$ such that

$$\int V S_{t_0}\mu \leq \gamma \int V\mu + K \int \mu \qquad \text{for all nonnegative } \mu \in \mathcal{M}. \quad (3)$$

Local Doeblin's condition

There exists $0 < \alpha < 1$ and $\nu \in \mathcal{P}$ with

$$S_{t_0}\mu \geq \alpha\nu \qquad \text{for all } \mu \in \mathcal{P} \text{ supported on a "large" set } \mathcal{C} \quad (4)$$

Then there exists a unique equilibrium $n_* \in \mathcal{P}$ and

$$\|S_t(\mu - \mu_*)\|_\beta \leq Ce^{-\lambda t}\|\mu - \mu_*\|_\beta. \qquad \text{for } t \geq 0 \qquad (5)$$

for some $\lambda > 0$, $C \geq 1$, $\beta > 0$ and all $\mu \in \mathcal{P}$.

Example: kinetic linear BGK

We consider

$$\partial_t f + v \cdot \nabla_x f = Lf,$$

posed for $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, where

$$Lf(x, v) := L^+ f(x, v) - f(x, v) := \left(\int_{\mathbb{R}^d} f(x, u) \, \mathrm{d}u \right) M(v) - f(x, v).$$

Call $T_t \equiv$ transport semigroup associated to $-v \nabla_x f$. Then:

$$f_t \geq e^{-t} \int_0^t \int_0^s T_{t-s} L^+ T_{s-r} L^+ T_r f_0 \, \mathrm{d}r \, \mathrm{d}s.$$

Two bounds

Jumping to small velocities

There exist $\alpha_L, \delta_L > 0$ such that for all x,

$$L^+ \delta_{(x,v)} \geq \alpha_L \, \delta_x \, \mathbb{1}[|v| \leq \delta_L].$$

Jumping to small positions

There exist $\alpha_T, \delta_T, t_0, \epsilon$ such that for all x,

$$\int_{\mathbb{R}^d} T_t\Big(\delta_x \mathbb{1}[|v| \leq \delta_L]\Big) \, \mathrm{d}v \geq \alpha_T \mathbb{1}[|x| \leq \delta_T] \qquad \forall t \in (t_0 - \epsilon, t_0 + \epsilon).$$

This allows us to use Doeblin's Theorem directly and obtain

$$\|f_t - M\|_1 \leq C e^{-\lambda t} \|f_0 - M\|_1 .$$

Example: kinetic linear BGK + potential

We consider

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi(x) \cdot \nabla_v f = Lf,$$

posed for $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, where

$$Lf(x,v) := L^+ f(x,v) - f(x,v) := \left(\int_{\mathbb{R}^d} f(x,u)\,\mathrm{d}u \right) M(v) - f(x,v).$$

Call $T_t \equiv$ semigroup associated to $-v\nabla_x f + \nabla_x \Phi(x) \cdot \nabla_v f$. Then:

$$f_t \geq e^{-t} \int_0^t \int_0^s T_{t-s} L^+ T_{s-r} L^+ T_r f_0 \,\mathrm{d}r\,\mathrm{d}s.$$

Two bounds (with potential Φ)

Jumping to small velocities

There exist $\alpha_L, \delta_L > 0$ such that

$$L^+ \delta_{(x,v)} \geq \alpha_L \, \delta_x \, \mathbb{1}[|v| \leq \delta_L].$$

Jumping to small positions

Assume $|\nabla \Phi(x)| \leq C |\Phi(x)|^\eta$ for some $0 < \eta < 1$. For any R_1 there exists δ_M, R_2, ϵ, t_0 such that

$$\int T_s(\delta_{x_0} \mathbb{1}[|v| \leq R_2]) \, \mathrm{d}v \geq \mathbb{1}[|x| \leq \delta_M] \quad \text{for } t \in [t_0 - \epsilon, t_0 + \epsilon]$$

for any $\Phi(x_0) \leq R_1$

We cannot obtain this for all x_0, so we need to use Harris instead of Doeblin.

The Foster-Lyapunov condition works with:

$$V(x,v) := 1 + \Phi(x) + \frac{1}{2}|v|^2 + \frac{1}{4} x \cdot v + \frac{1}{8}|x|^2$$

This allows us to use Harris' Theorem directly and obtain

$$\|f_t - M\|_* \leq C e^{-\lambda t} \|f_0 - M\|_*$$

with

$$\|g\|_* := \int |g| + \int V|g|.$$

Harris' theorem: proof

This proof is a simplification of that in **[Hairer & Mattingly 2011]**.

The norm $\|\cdot\|_\beta$ is here

$$\|\mu\|_\beta := \int |\mu| + \beta \int |\mu| V$$

It is enough to show that

$$\|S_{t_0}\mu\|_\beta \leq (1-\delta)\|\mu\|_\beta$$

for all measures μ with $\int \mu = 0$. (That is, S_{t_0} is contractive in the $\|\cdot\|_\beta$ norm). We extend our previous idea as follows:

First case: Contractivity for large V-moment.

Take $\delta \in (0, 1-\gamma)$. If we have

$$\int V|\mu| > \frac{K}{1-\gamma-\delta} \int |\mu| \tag{6}$$

then

$$\begin{aligned} \|S\mu\|_V = \int V|S\mu| &\leq \gamma \int V|\mu| + K \int |\mu| \\ &\leq (1-\delta) \int V|\mu| = (1-\delta)\|\mu\|_V. \end{aligned}$$

This easily implies the contractivity of the norm $\|\cdot\|_\beta$ in this case.

Assume that

$$\int V|\mu| < \frac{K}{1-\gamma-\delta}\int |\mu|, \tag{7}$$

In this case a sizeable part of the mass of μ_+ and μ_- is in $\mathcal{C}$:

$$\int_{\mathcal{C}} \mu_\pm \geq (1-\xi)\int \mu_\pm.$$

Using this we can carry out a similar argument as in Doeblin's Theorem: we have

$$S\mu_\pm \geq \alpha\nu \int_{\mathcal{C}} \mu_\pm \geq \alpha(1-\xi)\nu \int \mu_\pm =: \eta$$

and then

$$\|S\mu\| \leq \|S\mu_+ - \eta\| + \|S\mu_- - \eta\| = \int S\mu_+ + \int S\mu_- - 2\int \eta$$

$$= \int \mu_+ + \int \mu_- - \alpha(1-\xi)\int |\mu| = (1-\alpha(1-\xi))\int |\mu|.$$

It is then easy to complete the argument to obtain contractivity in the $\|\cdot\|_\beta$ norm, for β small enough:

$$
\begin{aligned}
\|S\mu\|_\beta &= \|S\mu\| + \beta \int V|S\mu| \\
&\leq (1 - \alpha' + \beta K)\|\mu\| + \beta\gamma \int V|\mu| \leq (1 - \delta_1)\|\mu\|_\beta,
\end{aligned}
$$

with

$$
\delta_1 := \min\{\alpha' - \beta K, 1 - \gamma\}. \tag{8}
$$

Taking then $\beta K < \alpha'$ gives the contractivity in this case.

Harris' theorem, subexponential version

$L \equiv$ generator of the semigroup.

Lyapunov condition

There exists $K \geq 0$ and functions $V \colon \Omega \to [1, +\infty)$, $\varphi \colon [1, +\infty) \to [1, +\infty)$ such that

$$\int VL\mu \leq -\int \varphi(V)\mu + K \int \mu \qquad \text{for all } \mu \in \mathcal{P}.$$

Local Doeblin's condition

There exists $0 < \alpha < 1$ and $\nu \in \mathcal{P}$ with

$$S_{t_0}\mu \geq \alpha\nu \qquad \text{for all } \mu \in \mathcal{P} \text{ supported on a "large" set } \mathcal{C} \tag{9}$$

Then there exists a unique equilibrium $n_* \in \mathcal{P}$ and

$$\|S_t(n - n_*)\| \leq r(t)\|n - n_*\|_V. \qquad \text{for } t \geq 0 \tag{10}$$

for some $r(t) \to 0$ as $t \to +\infty$ and all $n \in \mathcal{P}$.

We can in fact take

$$r(t) := \frac{1}{\varphi(H^{-1}(t))}$$

where

$$H(v) := \int_1^v \frac{1}{\varphi(u)}\,\mathrm{d}u.$$

Harris' theorem, subexponential version II

We can prove a slightly different version with the same technique: we add the condition:

Boundedness of moments

There exists $V_2 = V_2(x)$ with $V(x)/V_2(x) \to 0$ as $x \to +\infty$ such that

$$\int |S_t\mu| V_2 \leq C \int |\mu| V_2 \qquad \text{for all } t \geq 0,\ \mu \in \mathcal{P}.$$

Then we obtain the same conclusion with a different $r(t)$, which decays faster if V_2 is larger.

Subexponential version: proof

First case. *Estimate for large V-moment.* Assume that for $t \in [t_n, t_n + T]$ we have

$$\int V_0 |\mu_t| \geq \frac{2K}{c_1} \int |\mu_t|.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int V_1 |\mu_t| \leq -K \int V_0 |\mu_t|.$$

Consequently, if we choose any $\epsilon > 0$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int V_1 |\mu_t| &\leq -\epsilon \int V_1 |\mu_t| + c_1 \int_{V_0/V_1 \leq \epsilon/K} V_0 |\mu_t| \\
&\leq -\epsilon \int V_1 |\mu_t| + \epsilon_2 \int V_2 |\mu_t|,
\end{aligned}$$

where $\epsilon_2 \ll \epsilon$ as $\epsilon \to 0$. This implies

$$\int V_1 |\mu_{t_n+T}| \leq e^{-\epsilon T} \int V_1 |\mu_{t_n}| + T \epsilon_2 M_2$$

Subexponential version: proof II

Second case. *Estimate for small V-moment.* Assume now the contrary: there exists $t_* \in [t_n, t_n + T]$ such that

$$\int V|\mu_{t_*}| \leq \frac{2K}{c_1} \int |\mu_{t_*}|.$$

We take t_* to be the first time in $[t_n, t_n + T]$ that satisfies this. , and we set $t_{n+1} := t_* + T$. With the same bound as before, using the Doeblin assumption we get

$$\|\mu_{t_{n+1}}\|_\beta \leq e^{-\epsilon T} \|\mu_{t_*}\|_\beta + \beta T \epsilon_2 M_2.$$

Since $\epsilon_2 \ll \epsilon$ as $\epsilon \to 0$

One finally gets:

$$r(t) \approx (1 + M_2) \inf_{\epsilon > 0} \left\{ (1 - \epsilon T)^{t/t_0} + T\epsilon_2 \right\}.$$

A model for neuron populations

Proposed in **[Pakdaman, Perthame & Salort 2009]**:

$$\begin{cases} \dfrac{\partial}{\partial t} n(t,s) + \dfrac{\partial}{\partial s} n(t,s) + p(N(t),s)n(t,s) = 0, \quad t, s \geq 0, \\ N(t) := n(t, s=0) = \displaystyle\int_0^{+\infty} p(N(t),s)n(t,s)ds, \quad t > 0, \\ n(t=0,s) = n_0(s), \quad s > 0. \end{cases}$$

- $s \equiv$ time elapsed since last discharge
- $n(t,s) \equiv$ density of neurons which fired s seconds ago.
- $N(t) \equiv$ total activity at time t
- $p(N,s) \equiv$ probability of firing for a neuron that fired s seconds ago, given the total activity N.

The linear case

If p does not depend on N the equation becomes linear:

$$\begin{cases} \dfrac{\partial}{\partial t} n(t,s) + \dfrac{\partial}{\partial s} n(t,s) + p(s) n(t,s) = 0, \quad t, s \geq 0, \\ n(t, s=0) = \displaystyle\int_0^{+\infty} p(s) n(t,s) ds, \quad t > 0, \\ n(t=0, s) = n_0(s), \quad s > 0. \end{cases}$$

This generates a Markov semigroup in a space of measures (it's linear, conserves mass and positivity).

Notice that the entropy method cannot work in a straightforward way (but see works by **[Perthame & Ryzhik]**, **[Laurençot & Perthame]**, **[Pakdaman, Perthame & Salort]**).

One can apply directly Doeblin's theory to it **[Gabriel 2017]**.

How to show the Doeblin condition:

Step 1. n is larger than the solution $\tilde{n}$ to

$$\partial_t \tilde{n}(t,s) + \partial_s \tilde{n}(t,s) + p(s)\tilde{n}(t,s) = 0.$$

Step 2. n is larger than the solution to

$$\partial_t n(t,s) + \partial_s n(t,s) + p(s) n(t,s) = 0,$$

$$n(t,0) = \int_0^{+\infty} p(s)\tilde{n}(t,s)\,\mathrm{d}s.$$

The weakly nonlinear case

Main asumption: $|\partial_N p(N,s)|$ small.

Rewrite the nonlinear case as

$$\partial_t n = -\partial_s n - p(N,s)n + N\delta_0$$
$$= L_{N_*} n + (p(N_*,s) - p(N,s))n + \delta_0 \int_0^\infty (p(N,s) - p(N_*,s))n \,\mathrm{d}s$$
$$= L_{N_*} n + h$$

For a given equilibrium $n_* = n_*(s)$, call N_* the corresponding global activity, and L_{N_*} the operator

$$L_{N_*} n := -\partial_s n(t,s) - p(N_*,s)n(t,s) + \delta_0 \int_0^\infty p(N_*,s)n \,\mathrm{d}s$$

Then:

$$\partial_t(n - n_*) = L_{N_*}(n - n_*) + h$$

So

$$\partial_t(n - n_*) = L_{N_*}(n - n_*) + h,$$

or in other words,

$$n - n_* = S_t(n_0 - n_*) + \int_0^t S_{t-\tau} h(\tau)\,\mathrm{d}\tau$$

This allows us to get an exponential convergence close to n_*, since

$$\int h = 0, \qquad \|h\| \lesssim L\|n - n_*\|$$

A model for neuron populations with "fatigue"

Proposed in **[Pakdaman, Perthame & Salort 2014]**:

$$\begin{cases} \dfrac{\partial}{\partial t} n(t,s) + \dfrac{\partial}{\partial s} n(t,s) + p(N(t),s)n(t,s) \\ \qquad = \displaystyle\int_0^{+\infty} \kappa(s,u) p(N(t),u) n(t,u)\,du, \quad u,s,t \geq 0, \\ n(t,s=0) = 0, \quad N(t) = \displaystyle\int_0^{+\infty} p(N(t),s) n(t,s)\,ds, \\ n(t=0,s) = n_0(s) \geq 0, \quad \displaystyle\int_0^{+\infty} n_0(s)\,ds = 1. \end{cases}$$

- $s \equiv$ "state"
- $\kappa(s,u) \equiv$ probability distribution for the landing states when a neuron in state u jumps to a state s.

The linear case

$$\partial_t n(t,s)+\partial_s n(t,s)+p(N(t),s)n(t,s) = \int_0^{+\infty} \kappa(s,u)p(u)n(t,u)du.$$

How to show the Doeblin condition:
Step 1. n is larger than the solution $\tilde{n}$ to

$$\partial_t \tilde{n}(t,s) + \partial_s \tilde{n}(t,s) + p(s)\tilde{n}(t,s) = 0.$$

Step 2. n is larger than the solution to

$$\partial_t n(t,s) + \partial_s n(t,s) + p(s)n(t,s) = \int_0^{+\infty} \kappa(s,u)p(u)\tilde{n}(t,u)du.$$

We need a condition like:

$$\exists \epsilon, \delta > 0 : \quad \kappa(s,u) \geq \epsilon \mathbf{1}_{[0,\delta]}, \qquad \text{for all } s \geq s_*.$$

Thanks for listening!