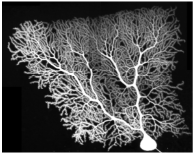
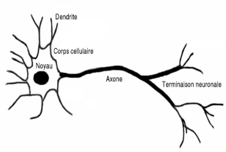
PDEs for neural networks

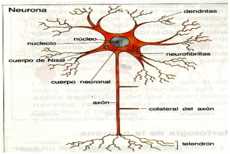
analysis and behaviour

Benoît Perthame

Course IV. Networks, time elapsed models







Plan of the course

I. The single neuron,

II. Networks, examples

III. Networks, Integrate & Fire

IV. Networks, time elapsed models

- **IV. 1.** time elapsed networks
- **IV. 2.** Blow-up and spontaneous activity
- **IV. 3.** Voltage vs time elapsed

Elapsed time structured model

Based on K. Pakdaman, J. Champagnat, J.-F. Vibert

- s represents the time elapsed since the last discharge
- $n(s,t)$ probability of finding a neuron in 'state' s at time t
- $N(t)$ = activity of the network

$$\frac{\partial n(s,t)}{\partial t} + \overbrace{\frac{\partial n(s,t)}{\partial s}}^{\text{elapsed time advances}} + \overbrace{r(s, bN(t))\ n(s,t)}^{\text{firing neurons}} = 0,$$

$$N(t) := n(s=0,t) = \underbrace{\int_0^{+\infty} r(s, bN(t))\ n(s,t)ds}_{\text{reset of firing neurons}},$$

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + r(s)\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

We want a steady state (compute the growth rate)

Long standing equation (Feller, renewal process)

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t)ds \end{cases}$$

We look for the first eigenfunction/eigenvalue

$$\begin{cases} \frac{\partial \bar{n}(s)}{\partial s} + [\lambda + r(s)]\ \bar{n}(s) = 0 \\ \bar{n}(s=0) = \int_0^{+\infty} b(s)\ \bar{n}(s)ds \end{cases}$$

$$\bar{n}(s) > 0$$

The renewal equation

We look for the first eigenfunction/eigenvalue

$$\begin{cases} \frac{\partial \bar{n}(s)}{\partial s} + [\lambda + r(s)]\ \bar{n}(s) = 0, \\ \bar{n}(s=0) = \int_0^{+\infty} b(s)\ \bar{n}(s) ds \end{cases}$$

$$\bar{n}(s) = \bar{n}(0) e^{-R(s)-\lambda s}, \qquad R(s) = \int_0^s r$$

In the boundary condition

$$\bar{n}(0) = \bar{n}(0) \int_0^\infty b(s) e^{-R(s)-\lambda s} ds$$

Find λ such that

$$\int_0^\infty b(s) e^{-R(s)-\lambda s} ds = 1$$

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

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Find λ such that

$$\int_0^{\infty} b(s) e^{-R(s)-\lambda s} ds = 1$$

When $b(\cdot) = r(\cdot)$ then $\lambda = 0$

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

$$\bar{n}(s) = \bar{n}(0) e^{-R(s)-\lambda s}, \qquad \int_0^{\infty} b(s) e^{-R(s)-\lambda s} ds = 1$$

Prove that

$$n(s,t) \to \bar{\rho}\bar{n}(s)$$

Use Generalized Relative Entropy

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

$$n(s,t) \to \bar{\rho}\bar{n}(s)$$

Dual eigenfunction

$$\frac{\partial \phi(s)}{\partial s} - [\lambda + r(s)]\ \phi(s) = -\phi(0) b(s)$$

We have

$$\frac{d}{dt} \int n(s,t)\phi(s) ds = 0$$

$$\int n(s,t=0)\phi(s) ds = \bar{\rho} \int \bar{n}(s)\phi(s) ds$$

So the dual eigenfunction is important

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

Dual eigenfunction

$$\frac{\partial \phi(s)}{\partial s} - [\lambda + r(s)]\ \phi(s) = -\phi(0) b(s)$$

Usually for a conservative equation $\phi \equiv 1$

- For example, when $b(\cdot) = r(\cdot)$, then $\lambda = 0$.
- Then the relative entropy is $\int_0^{\infty} \bar{n}(s) H(\frac{n(s,t)}{\bar{n}(s)}) ds$
- Other standard example is the Fokker-Planck equation

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

$$\frac{\partial \phi(s)}{\partial s} - [\lambda + r(s)]\ \phi(s) = -\phi(0) b(s)$$

The Generalized Relative Entropy is, with $u(s,t) = \frac{n(s,t)e^{-\lambda t}}{\bar{n}(s)}$

$$E(t) := \int_0^{\infty} \phi(s)\bar{n}(s) H(u(s,t)) ds$$

The renewal equation

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda + r(s)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s)\ n(s,t) ds \end{cases}$$

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The Generalized Relative Entropy is, with $u(s,t) = \frac{n(s,t)e^{-\lambda t}}{\bar{n}(s)}$

$$E(t) := \int_0^\infty \phi(s)\bar{n}(s) H(u(s,t)) ds$$

$$\frac{dE(t)}{dt} = -D_H(t)\phi(0)$$

$$D_H(t) = \int b(s)\bar{n}(s) H\left(\frac{n(s,t)}{\bar{n}(s)}\right) ds - H\Big(\int b(s)\bar{n}(s)\frac{n(s,t)}{\bar{n}(s)} ds\Big)$$

> 0 by Jensen inequality

The renewal equation

And a Poincaré inequality ?

$$\nu \int_0^\infty \phi(s)\bar{n}(s)|u(s)|ds \leq \int b(s)\bar{n}(s)|u(s)|ds - \left| \int_0^\infty b(s)\bar{n}(s)u(s)ds \right|$$

when

$$\int b(s)\bar{n}(s) = 1, \qquad \int u(s)\bar{n}(s)\phi(s)ds = 0$$

Theorem

If $b(\cdot) \geq \nu\phi(s)$, then the Poincaré inequality holds true.

The renewal equation

Poincaré inequality

$$\nu \int_0^\infty \phi(s)\bar{n}(s)|u(s)|ds \leq \int b(s)\bar{n}(s)|u(s)|ds - \Big| \int_0^\infty b(s)\bar{n}(s)u(s)ds \Big|$$

when

$$\int b(s)\bar{n}(s) = 1, \qquad \int u(s)\bar{n}(s)\phi(s)ds = 0$$

Proof of Poincaré inequality

$$\Big| \int_0^\infty b(s)\bar{n}(s)u(s)ds \Big| = \Big| \int_0^\infty [b(s) - \nu\phi(s)]\bar{n}(s)u(s)ds \Big|$$

The renewal equation

Poincaré inequality

$$\nu \int_0^\infty \phi(s)\bar{n}(s)|u(s)|ds \leq \int b(s)\bar{n}(s)|u(s)|ds - \Big| \int_0^\infty b(s)\bar{n}(s)u(s)ds \Big|$$

when

$$\int b(s)\bar{n}(s) = 1, \qquad \int u(s)\bar{n}(s)\phi(s)ds = 0$$

Proof of Poincaré inequality

$$\Big| \int_0^\infty b(s)\bar{n}(s)u(s)ds \Big| = \Big| \int_0^\infty [b(s) - \nu\phi(s)]\bar{n}(s)u(s)ds \Big|$$

$$\geq \int_0^\infty [b(s) - \nu\phi(s)]\bar{n}(s)|u(s)|ds$$

The renewal equation

Poincaré inequality

$$\nu \int_0^\infty \phi(s)\bar{n}(s)|u(s)|ds \le \int b(s)\bar{n}(s)|u(s)|ds - \left| \int_0^\infty b(s)\bar{n}(s)u(s)ds \right|$$

when

$$\int b(s)\bar{n}(s) = 1, \qquad \int u(s)\bar{n}(s)\phi(s)ds = 0$$

Proof of Poincaré inequality

$$\left| \int_0^\infty b(s)\bar{n}(s)u(s)ds \right| = \left| \int_0^\infty [b(s) - \nu\phi(s)]\bar{n}(s)u(s)ds \right|$$

$$\ge \int_0^\infty [b(s) - \nu\phi(s)]\bar{n}(s)|u(s)|ds$$

$$= \int_0^\infty b(s)\bar{n}(s)|u(s)|ds - \int_0^\infty \nu\phi(s)\bar{n}(s)|u(s)|ds$$

Floquet case

The GRE is robust

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + r(s,t)\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s,t)\ n(s,t) ds \end{cases}$$

Assume that $r(s,t),\ b(s,t)$ are time periodic

Floquet case

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda_F + r(s,t)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s,t)\ n(s,t) ds \end{cases}$$

$r(s,t),\ b(s,t)$ are time periodic

Floquet case

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda_F + r(s,t)]\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s,t)\ n(s,t) ds \end{cases}$$

$r(s,t),\ b(s,t)$ are time periodic

$$\begin{cases} \frac{\partial \bar{n}(s,t)}{\partial t} + \frac{\partial \bar{n}(s,t)}{\partial s} + [\lambda_F + r(s,t)]\ \bar{n}(s,t) = 0, \\ \bar{N}(t) := \bar{n}(s=0,t) = \int_0^{+\infty} b(s,t)\ \bar{n}(s,t) ds \end{cases}$$

$$\frac{\partial \bar{\phi}(s,t)}{\partial t} + \frac{\partial \bar{\phi}(s,t)}{\partial s} - [\lambda_F + r(s,t)]\ \bar{\phi}(s,t) = \phi(0,t) b(s,t),$$

with

$$\bar{n}(s,t) > 0,\ \bar{\phi}(s,t) > 0 \text{ time periodic}$$

Floquet case

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + [\lambda_F + r(s,t)] \; n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} b(s,t) \; n(s,t) ds \end{cases}$$

The Generalized Relative Entropy is

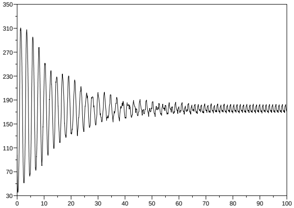
$$E(t) := \int_0^{\infty} \bar{\phi}(s,t) \bar{n}(s,t) H(\frac{n(s,t)}{\bar{n}(s,t)}) ds$$

$$\frac{dE(t)}{dt} = -D_H(t) \phi(0,t)$$

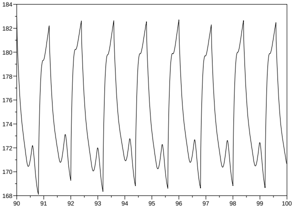
$$D_H(t) = \int b(s,t) \bar{n}(s,t) H\left(\frac{n(s,t)}{\bar{n}(s,t)}\right) ds - H\Big(\int b(s,t) \bar{n}(s,t) \frac{n(s,t)}{\bar{n}(s,t)} ds\Big)$$

$$> 0 \qquad \text{by Jensen inequality}$$

Floquet case



global simulation



zoom on the last periods

Total cell density $\int_0^\infty n(s,t)ds$ as a function of time

Elapsed time structured model

$$\frac{\partial n(s,t)}{\partial t} + \overbrace{\frac{\partial n(s,t)}{\partial s}}^{\text{elapsed time advances}} + \overbrace{r(s, bN(t))\ n(s,t)}^{\text{firing neurons}} = 0,$$

$$N(t) := n(s=0,t) = \underbrace{\int_0^{+\infty} r(s, bN(t))\ n(s,t) ds,}_{\text{reset of firing neurons}}$$

Theorem For small or large connectivity ($b > 0$ small or large) then desynchronization still holds

$$n(s,t) \underset{t\to\infty}{\longrightarrow} P_b(s)$$

Elapsed time structured model

$$\frac{\partial n(s,t)}{\partial t} + \overbrace{\frac{\partial n(s,t)}{\partial s}}^{\text{elapsed time advances}} + \overbrace{r(s, bN(t))\ n(s,t)}^{\text{firing neurons}} = 0,$$

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Theorem For small or large connectivity $(b > 0$ small or large$)$ then desynchronization still holds

$$n(s,t) \underset{t\to\infty}{\longrightarrow} P_b(s)$$

• In the middle range connectivity there are several periodic solutions (analytic forms of solutions),

• These are stable (observed numerically).

Elapsed time structured model

$$\frac{\partial n(s,t)}{\partial t} + \overbrace{\frac{\partial n(s,t)}{\partial s}}^{\text{elapsed time advances}} + \overbrace{r(s, bN(t))\ n(s,t)}^{\text{firing neurons}} = 0,$$

$$N(t) := n(s=0,t) = \underbrace{\int_0^{+\infty} r(s, bN(t))\ n(s,t) ds,}_{\text{reset of firing neurons}}$$

Theorem For small or large connectivity ($b > 0$ small or large) then desynchronization still holds

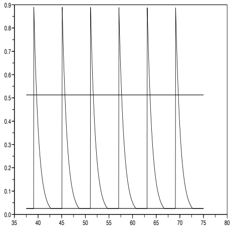
$$n(s,t) \underset{t \to \infty}{\longrightarrow} P_b(s)$$

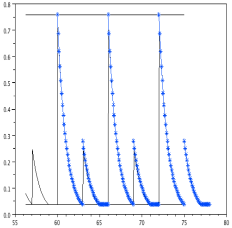
- Most accurate results today : Mischler, Cañizo-Yoldas

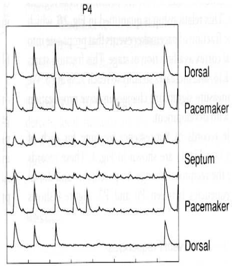
Elapsed time structured model

$$\frac{\partial n(s,t)}{\partial t} + \overbrace{\frac{\partial n(s,t)}{\partial s}}^{\text{elapsed time advances}} + \overbrace{r(s, bN(t))\ n(s,t)}^{\text{firing neurons}} = 0,$$

$$N(t) := n(s=0,t) = \underbrace{\int_0^{+\infty} r(s, bN(t))\ n(s,t) ds,}_{\text{reset of firing neurons}}$$

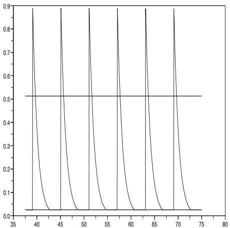


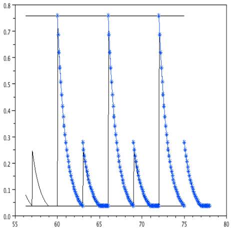


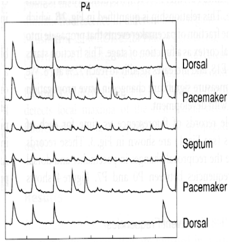


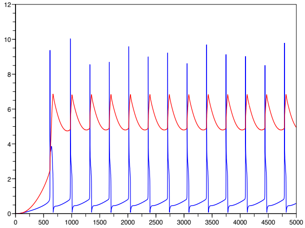
Right : Conhaim et al (2011) J. of physiology 589(10) 2529-2541.

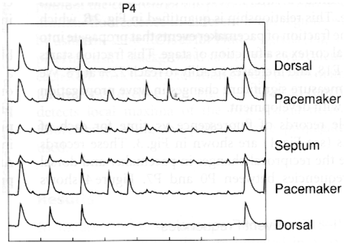
Comparison with I&F











Elapsed time and IF

We encountered two equations

Time elapsed

$$\begin{cases} \partial_t n + \partial_s n + r(s) n = 0, \\ n(s=0,t) = \int_0^\infty r(s) n(s,t) ds =: N(t), \\ n(s,0) = n^0(s), \end{cases}$$

LIF

$$\begin{cases} \partial_t \widehat{n} + \partial_v [h(v)\widehat{n}] - a\partial^2_{vv}\widehat{n} = N(t)\delta(v - V_R), \\ \widehat{n}(V_F, t) = 0, \qquad N(t) := -a\partial_v \widehat{n}(V_F, t). \end{cases}$$

Is there a connection between these two equations ?

Elapsed time and IF

From G. Dumont and J. Henry (linear equations)

Being given V_R, V_F, $h(\cdot)$ and

$$\begin{cases} \partial_t \hat{n} + \partial_v [h(v)\hat{n}] - a\partial^2_{vv}\hat{n} = N(t)\delta(v - V_R), \\ \hat{n}(V_F, t) = 0, \qquad -a\partial_v \hat{n}(V_F, t) = N(t). \end{cases}$$

Theorem There are functions $r(s)$, $q(v, s)$ such that, for solutions of

$$\begin{cases} \partial_t n + \partial_s n + r(s)n = 0, \\ n(s = 0, t) = \int_0^\infty r(s)n(s, t)ds =: N(t), \\ n(s, 0) = n^0(s), \end{cases}$$

then, with the same $N(t)$, we have

$$\hat{n}(v, t) = \int_0^\infty q(v, s)n(s, t)ds$$

Elapsed time and IF

$$\begin{cases} \partial_s q(v,s) + \partial_v [h(v)q] - \partial^2_{vv} q = r(s)q \\ q(V_F, s) = 0 \\ q(v, s=0) = \delta(v - V_R) \\ r(s) := -\partial_v q(V_F, s) \qquad \qquad (implicit) \end{cases}$$

We notice that $\int_{-\infty}^{V_F} q(v,s)dv = 1, \qquad \forall s \geq 0.$

Because

$$\frac{d}{ds} \int_{-\infty}^{V_F} q(v,s)dv + r(s) = r(s) \int_{-\infty}^{V_F} q(v,s)dv.$$

Next, let $n(s,t)$ solve the age structure equation

$$\begin{cases} \partial_t n + \partial_s n + r(s) n = 0, \\ n(s=0,t) = \int_0^\infty r(s) n(s,t) ds =: N(t), \\ n(s,0) = n^0(s), \qquad \text{with} \qquad \int_0^\infty n^0(s) ds = 1. \end{cases}$$

Elapsed time and IF

Is there a solution to

$$\begin{cases} \partial_s q(v,s) + \partial_v[h(v)q] - \partial^2_{vv} q = r(s)q \\ q(V_F, s) = 0 \\ q(v, s=0) = \delta(v - V_R) \\ r(s) := -\partial_v q(V_F, s) \qquad \qquad (implicit) \end{cases}$$

Elapsed time and IF

Is there a solution to

$$\begin{cases} \partial_s q(v,s) + \partial_v[h(v)q] - \partial^2_{vv} q = r(s)q \\ q(V_F, s) = 0 \\ q(v, s=0) = \delta(v - V_R) \\ r(s) := -\partial_v q(V_F, s) \qquad (implicit) \end{cases}$$

$$\begin{cases} \partial_s q_0(v,s) + \partial_v[h(v)q_0] - \partial^2_{vv} q_0 = 0 \\ q_0(V_F, s) = 0 \\ q_0(v, s=0) = \delta(v - V_R) \end{cases}$$

and set

$$q(v,s) = \frac{q_0(v,s)}{\int q_0(v,s)dv}$$

Networks of networks

$$\begin{cases} \partial_t n(s,x,t) + \partial_s n + r(s,S(t,x))n = 0, \\ N(x,t) := n(s=0,x,t) = \int_0^\infty r(s)n(s,x,t)ds \end{cases}$$

$$\frac{dS(t,x)}{dt} + S(t,x) = \int w(x,y)N(t,y)dy$$

Networks of networks

$$\begin{cases} \partial_t n(s,x,t) + \partial_s n + r(s, S(t,x))n = 0, \\ N(x,t) := n(s=0,x,t) = \int_0^\infty r(s) n(s,x,t) ds \end{cases}$$

$$\frac{dS(t,x)}{dt} + S(t,x) = \int w(x,y) N(t,y) dy$$

Assume quasi-steady state

$$n(s,x) = n(0,x) \exp(R(s,x,\mathcal{N})), \qquad \int n(s,x) ds = 1$$

$$N(x,t) = \Big(\int \exp(-R(s,S(x,t)) ds \Big)^{-1}$$

Networks of networks

$$\begin{cases} \partial_t n(s,x,t) + \partial_s n + r(s, S(x,t))n = 0, \\ N(x,t) := n(s=0,x,t) = \int_0^\infty r(s)n(s,x,t)ds \end{cases}$$

$$\frac{dS(t,x)}{dt} + S(t,x) = \int w(x,y)N(y,t)dy$$

Assume quasi-steady state

$$n(s,x) = n(0,x)\exp(R(s,x,\mathcal{N})), \qquad \int n(s,x)ds = 1$$

$$N(x,t) = \Big(\int \exp(-R(s,S(x,t))ds\Big)^{-1}$$

One gets Wilson-Cowan type model

$$\frac{dS(x,t)}{dt} + S(x,t) = \int w(x,y)\sigma\big(S(y,t)\big)dy$$

CONCLUSION

THANKS TO MY COLLABORATORS

M. J. Carceres

J. A. Carrillo

D. Smets

D. Salort

K. Pakdaman

C. Wainrib

CONCLUSION

THANKS TO MY COLLABORATORS

M. J. Carceres, J. A. Carrillo

D. Smets, D. Salort

K. Pakdaman

THANK YOU