Hessian Estimates for Schrödinger Equations

Xue-Mei Li
Imperial College London

XIX school of Mathematics Louis Santaló
Interaction between PDE and Probability
13-17 August 2018

Motivation

Analysis of loop and path spaces over manifold

They are Banach manifolds,with measures from BM or Brownian bridge. Gross, Segal, Aida, Airault, Andersson, Bakry, Bismut, Cruzeiro, Driver, Elworthy, Fang, Franchi, Funaki, Gong, Hsu, Ikeda, Kusuoka, Ledoux, xml, Ma, Shigekawa, Sugita, Stroock, Wang, Watanabe, ... and global analysts working smooth loops.

- Study its topology by establishing an de Rham complex, and deRham cohomology, for which needs an L_2 Hodge theory.
- We need Hilbert subspaces of the tangent spaces and their tensor spaces. This are chosen by Cameron-Martin's invariant theorem and are finite 'energy' spaces.
- To define global charts and a Sobolev calculus.
- study the BM measure: concentration and tail? Poincare, Logarithmic Sobolev inequality? Need heat kernel are Euclidean like up to 2 derivatives.
- Poincare on loop space fails on some simple compact manifolds (Eberle). Wiener space (Gross'67), Asymptotically Euclidean: Aida, Hyperbolic space: Chen-xml-Wu (2012)

A solvable Schrödinger Equation

$$\begin{cases} \dfrac{\partial g}{\partial t} & = & \displaystyle\sum_{i=1}^{n} \frac{\partial^2 g}{\partial x_i^x} - x \cdot \nabla g - C\, g \\ g(0,x) & = & f(x) \end{cases}$$

Then

$$g(t,\cdot) = e^{-Ct} Q_t f,$$

where Q_t is the Ornstein-Uhlenbeck semi-group, given by Mehler transform:

$$Q_t f(x) = \int f(z)\, \overbrace{\frac{\exp\left\{-\frac{|z-e^{-t}x|^2}{2(1-e^{-2t})}\right\}}{(2\pi(1-e^{-2t}))^{n/2}}}^{q^V(t,x,z)}\, dz,$$

The continuous Talagrand conjecture

Hyper-contractivity: Nelson 60's, Gross 75: for $t > 0$, $p > 1$ and $q \leq 1 + (p-1)e^{2t}$,

$$|Q_t g|_q \leq |g|_p.$$

For $p = 1$, Talagrand conjectured (for a discrete system)

$$\lim_{A \to \infty} \sup_{g \geq 0,\, g \in L^1(\gamma)} A\, \gamma\left(\{x : Q_t g(x) \geq A\}\right) = 0, \quad t > 0.$$

Lehec16, Ball-Barthe-Bednorz-Oleszkiewicz-Wolff 13, Eldan-Lee14:

$$\gamma\left(\{x : Q_s g(x) \geq A\}\right) \leq \alpha \frac{1 \vee \frac{1}{2s}}{A\sqrt{\log A}}.$$

whose proof relies on the following estimate:

$$\mathrm{Hess}(\log Q_t g) \geq -\frac{1}{2t} Id, \qquad g \in L_1(\gamma).$$

Question : Does this hold if the potential $U(x) = \frac{1}{2}|x|^2$ is replaced by a function h, (a perturbation)?

The perturbed Ornstein-Uhlenbeck semigroup

Let h be a function. Set $W(x) = \frac{|x|^2}{2} - h(x)$ and

$$V = \frac{1}{2}(n - \Delta h) - \frac{1}{4}(|x|^2 - |\nabla h|^2).$$

Consider the semigroup P_t^h generated by

$$\Delta - \nabla h \cdot \nabla$$

and Q_t^V for

$$\Delta - x \cdot \nabla - V$$

Then they are unitarily equivalent:

$$P_t^h = e^{-W/2} Q_t^V (e^{W/2} f).$$

In [Gozlan, xml, Madiman, Roberto, Samson, 18+], we use this and a Brownian-bridge to solve the conjecture for perturbed O-U semigroup.

Schrödinger Equation and diffusion operators

Schrödinger Equation

$$\begin{cases} \frac{\partial g}{\partial t} & = & (\mathcal{L} - V)g, \\ g(0,x) & = & f(x) \end{cases}$$

Where $\mathcal{L}$ is a diffusion operator, e.g.

$$\mathcal{L} = \frac{1}{2}\Delta + \nabla h.$$

If $\mathcal{L}$ is elliptic and V is bounded Hölder continuous, then there exists a semigroup P_t^V on $L_2 \cap L_\infty$ such that

$$g(x,y) \equiv P_t^V f(x).$$

And there exists an integral kernel:

$$P_t^V f(x) = \int f(z) p^V(t,x,z)\, dz.$$

Problem. Gradient Estimates, Hessian Estimates for solutions and the logarithms of the kernels.

Diffusion Operators

In local coordinates, diffusion operators are of the form

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k(x)\frac{\partial}{\partial x_k}.$$

$A(x) := (a_{i,j}(x))$ is a $n \times n$ symmetric non-negative matrix. If Lipschitz continuous, $A(x) = X^T(x)X(x)$. The columns of the $n \times m$ matrix X represents vector fields $X_1, \dots, X_m$, $m \geq n$. Think: $X(x) \in L(\mathbf{R}^m; T_x M)$. Then

$$\mathcal{L}f = \frac{1}{2}\sum X_k(X_k f) + X_0 f.$$

The drift can always be removed so $\mathcal{L} = \sum (Y_i)^2$.

- $\mathcal{L}$ is **elliptic** if $A(x)$ is strictly positive at every point, equivalently $X(x) : \mathbf{R}^m \to T_x M$ is a surjection and so determines a Riemannian metric.
- $\mathcal{L}$ is **semi-elliptic** if

$$E_x := \mathrm{span}\{X_1(x), \dots, X_m(x)\}$$

Connections induced by diffusion operators

Theorem [Elworthy-LeJan-xml 97,99] If $\mathcal{L}$ is semi-elliptic, $X(x)Y(x) = T_xM$. For any vector field $Z \in \Gamma E$ and for any $v \in T_xM$, we define

$$\nabla_v Z = X(x) d\left(Y(\cdot)Z(\cdot)\right)(v_0).$$

- Then ∇ defines a linear semi-connection (a set of Christoffel symbols), with possibly torsion.
- ∇ is uniquely determined by the property that

$$\nabla X(e) \equiv 0, \quad e \in [\ker X(x)]^{\perp}.$$

- Every metric semi-connection is induced by a map X.
- Nice property on Derivative flow $dV_t = \nabla X(V_t) \circ dB_t + \nabla X_0(V_t)dt$.
- extended to equi-variant diffusion pairs (Elworthy-LeJan-xml 2008]: useful for **slow-fast dynamics.**

- **The Gradient system.** If $\phi = (\phi_1, \dots, \phi_m) : M \to \mathbf{R}^n$ is an isometric embedding of M, we set

$$X_i(x) = \nabla \phi_i.$$

Then X defines the Levi-Civita connection. Also,

$$\sum_i (X_i)^2 = \Delta.$$

- Given $\mathcal{L} = \frac{1}{2} \sum (X_i)^2 + X_0$, define

$$dx_t = \sum_{i=1}^{m} X_i(x_t) \circ dB_t^i + X_0(x_t)dt.$$

The solutions are strong **Markov process with generator $\mathcal{L}$.**

- Definition: A strong Markov process with generator $\frac{1}{2}\Delta$ is a **Brownian motion.**

BM and heat Equation

Suppose that there is no explosion to

$$dx_t = \sum_{i=1}^{m} X_i(x_t) \circ dB_t^i + X_0(x_t)dt = X(x_t) \circ dB_t + X_0(x_t)dt.$$

By $\mathcal{L}$ is the generator, we meant, roughly speaking, that $\mathbf{E}f(x_t)$ solves the equation

$$\frac{\partial g}{\partial t} = \mathcal{L}g.$$

Proof. Let $P_t f$ denote the solution of the heat equation with initial value f. By Itô's formula:

$$f(x_t) = P_t f(x_0) + \int_0^t dP_{T-s} f(x_s)(X(x_s)dB_s).$$

Taking expectation to see the claim.

A BEL martingale technique

Introducing a technique

Assume suitable growth conditions on X_i and on their derivatives. **BEL formula** holds for any bounded Borel measurable function f and $v_0 \in T_{x_0}M$:

$$dP_t f(v_0) = \frac{1}{t}\mathbf{E}\left[f(x_t)\int_0^t \langle W_s(v_0), X(x_s)dB_s\rangle\right],$$

where $W_s : T_{x_0}M \to T_{x_s}M$ is the damped stochastic parallel transport or the derivative flow.

Proof. Multiply by $\int_0^t \langle W_s(v_0), X(x_s)dB_s\rangle$

$$f(x_t) = P_t f(x_0) + \int_0^t dP_{T-s}f(x_s)(X(x_s)dB_s).$$

$$\mathbf{E}\left(f(x_t)\int_0^t \langle W_s(v_0), X(x_s)dB_s\rangle\right) = \int_0^t \mathbf{E}\left(dP_{t-s}f(x_s)(W_s(v_0))\right) ds$$

$$= \int_0^t dP_t f(v_0)\, ds = t\; dP_t f(v_0).$$

[xml92, Elworthy-xml, c.f. Bismut]

Damped Parallel translation

- We used a result of Airault

$$dP_t f(v) = \mathbf{E} df(W_t(v_0))$$

  and Markov property on forms.

- The damped parallel translation $W_t : T_{x_0 M} \to T_{x_t} M$ solves:

$$\frac{D}{dt} W_t = \frac{1}{2}\left(-\mathrm{Ric}^{\sharp}_{x_t}\right)(W_t), \quad W_0 = \mathrm{id}_{T_{x_0} M}$$

  Elworthy, Eells, Malliavin,..

  Proof. Apply Itô's formula to $dP_{t-r} f(W_r(v_0)$. Observe :

$$\frac{\partial}{\partial t} df_t = -\frac{1}{2}\nabla^* \nabla df_t - df_t(\frac{1}{2}\mathrm{Ric}^{\sharp} -) :$$

  Then,

$$df(W_t(v_0)) = dP_t f(W_t(v_0)) + \mathit{martingale}.$$

Remark

1. The formula implies strong Feller property, uniqueness of the invariant probability measure. Hairer-Mattingly, ...
2. BEL formula is equivalent to Integration by part formula holds for the measure of the Markov process on the path space (Wiener measure.) Clark-Ocone fomula, Logarithmic Sobolev inequality, Poincare inequality. Aida, Bismut, Elworthy, Driver, Fang, Hsu, LeJan, xml, Malliavin, Norris, Shigekawa,
3. Method of proof extends to
    - Dirichlet and Neunman boundary conditions and local: Arnaudon, Deuschel, xml, Thalmaier, Wang, Zambotti,
    - semi-elliptic case: Elworthy, LeJan, xml
    - SPDE: Elworthy, Da Prato, Zabczyck, ...
    - jumps: Takeuchi, Kateregga, Zabczyk, Peszat, Da Prato, ..
    - mean field PDE: .D. Banos
    - numerical, simulation irregular dritfs, finance: Henry-Labordere, Tang, Touzi, Caas Friz, Bayer, Dong, Xu
    - higher order derivatives (caution): Elworthy, xml, Malliaivin, Stroock,

Application to Reaction-Diffusion equations

$$\begin{cases} \dfrac{\partial u_t^\epsilon(x)}{\partial t} = \dfrac{\epsilon^2}{2}\Delta u_t^\epsilon(x) + \dfrac{1}{\epsilon^2} c(t,x,u_t^\epsilon(x)) u_t^\epsilon(x), \\ u^\epsilon(0,x) = T_0(x)\exp\{-\dfrac{S_0(x)}{\epsilon^2}\}. \end{cases}$$

For example take $c = 1 - u$, Freidlin showed that the solutions converges to a travelling waves. : there exists $V(t,x)$ such that the limit is 1 when $V(t,x) > 0$ and equals 0 if $V(t,x) < 0$. In fact the derivatives converges to 0 at the trough exponentially fast. [xml-Zhao96.]

Heat kernel estimates
(a quick review)

Examples

On $\mathbf{R}^n$, $p_t(x,y) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2t}}$.

$$\nabla \log p_t = -\frac{x-y}{t}, \quad \nabla^2 \log p_t(u,v) = -\frac{\langle u, v\rangle}{t}.$$

$$\nabla^2 \log p_t(u,v) + \frac{\langle u, v\rangle}{t} = 0.$$

Messy in general, even on the hyperbolic space, $r = d(x,y)$,

Geometers typically work on Gaussian and gradient or derivative for positive solutions: Cheeger, Li-Yau estimates, Mckean, Grigoryan, Davies, Cheng, Gromov, Taylor,...

Stochastic analysts are interested in logarithm of the heat kernel: Varadhan, Azencott, Molchanov, Aida, Airault, Hsu, Sheu, Malliavin, Stroock,...

Kernel estimates

How does the integral kernel (heat kernel)behave? If $\mathcal{L}$ is strictly elliptic with bounds λ_1 and λ_2, then the kernel is controlled by that for $\lambda_i \Delta$.
The gradient of the kernel cannot be bounded as such. They depend on differentiating $a_{i,j}$. Treating $(a_{i,j})$ as the inverse of the Riemannian tensor, we want the dependence on the derivatives of the Riemannian metric only through the Ricci curvature or its derivatives. This is where the methods /results depart from Euclidean methods.

$$Ric_{i,j} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \Gamma^m_{il}\Gamma^l_{km} - \nabla_k \left(\frac{\partial}{\partial x^l} \log \sqrt{|g|} \right).$$

$$\Gamma^k_{i,j} = \frac{1}{2} r^{m,k} \left(g_{k,i}, j + g_{kj,i} - g_{i,.j}, k \right).$$

These quantities are a mess in terms of $a_{i,j}$.

Varadhan Estimates

Varadhan, Molchanov, Azencott et al. Suppose $K \subset M$ is a compact subset of M, then

$$\lim_{t \downarrow 0} t \log p(t, x, y) = -\frac{d_M(x, y)^2}{2}, \ \ \forall\, x, y \in M, \qquad (0.1)$$

and the convergence is uniformly for $(x, y) \in K \times K$.
Moreover, for every connected bounded open set $D \supseteq K$ with smooth boundary,

$$\lim_{t \downarrow 0} t \log \mathbb{P}_x\left(\tau_D > t\right) = -\frac{1}{2} d_M(x, \partial D)^2, \ \ \forall\, x \in K, \qquad (0.2)$$

and the convergence is uniformly for $x \in K$ Here $\tau_D(\gamma) := \inf\{t > 0; \gamma(t) \notin D\}$, $d_M(x, \partial D) := \inf_{z \in \partial D} d_M(x, z)$.

Small time

The following statements hold [Chen-xml-Wu 18+]

(1) Suppose $x, y \in M$ and $x \notin \mathrm{Cut}_M(y)$, then

$$\lim_{t \downarrow 0} t \nabla_x \log p(t, x, y) = -\nabla_x \left(\frac{d_M^2(x, y)}{2} \right). \tag{0.3}$$

Here the convergence is uniformly for $x \in \tilde{K}$ with $\tilde{K}$ being a compact subset of $\mathrm{Cut}_M^c(y)$.

(2) Suppose $K \subset M$ is a compact subset of M, then there exists a positive constant $C_1(K)$, (which depends on K) such that

$$|\nabla_x \log p(t, x, y)|_{T_x M} \leq C_1 \left(\frac{d_M(x, y)}{t} + \frac{1}{\sqrt{t}} \right), \ x, y \in K, \ t \in (0, 1]. \tag{0.4}$$

Use a perturbation result on the canonical SDE on the orthonormal frame bundle.

Small time Hessian heat kernel/ no curvature conditions

Assume complete and stochastic complete. The following statements hold [[Chen-xml-Wu 18+]

(1) Suppose $y \in M$ and $\tilde{K} \subset \mathrm{Cut}_M^c(y)$ is a compact set, then

$$\limsup_{t \downarrow 0} \sup_{x \in \tilde{K}} \left| t\nabla_x^2 \log p(t,x,y) + \nabla_x^2 \left(\frac{d_M^2(x,y)}{2} \right) \right|_{T_xM \otimes T_xM} = 0. \quad (0.5)$$

(2) For every $y \in M$ and $r_0 \in (0, \mathrm{inj}(y))$, there exist positive constants $t_0(y, r_0)$ and $C_1(y, r_0)$, such that

$$\left| t\nabla_x^2 \log p(t,x,y) + \mathbf{I}_{T_xM} \right|_{T_xM \otimes T_xM} \leq C_1 \left(d_M(x,y) + \sqrt{t} \right), \quad x \in B_y \quad (0.6)$$

where $\mathbf{I}_{T_xM}$ is identical map on T_xM.

(3) Suppose $K \subset M$ is a compact subset of M, then there exists a positive constant $C_2(K)$, such that

$$\left| \nabla_x^2 \log p(t,x,y) \right|_{T_xM \otimes T_xM} \leq C_2 \left(\frac{d_M^2(x,y)}{t^2} + \frac{1}{t} \right), \ x, y \in K, \ t \in (0$$

(0.7)

Crucial lemma 1

For any $m \in \mathbb{N}$, there exists a stochastic process(vector fields) $l_m : [0,1] \times C([0,1]; M) \to [0,1]$, such that [Chen-xml-Wu 18+]

(1) $l_m(t,\gamma) = \begin{cases} 1, & t \le \tau_{m-1}(\gamma) \wedge 1 \\ 0, & t > \tau_m(\gamma) \end{cases}$.

(2) Given any $x \in D_m$, $l_m(t,\gamma)$ is $\mathcal{F}_t^\gamma := \sigma\{\gamma(s); s \in [0,t]\}$-adapted and $l_m(\cdot,\gamma)$ is absolutely continuous for μ_x-a.s. $\gamma \in P_x(M)$.

(3) For every positive integer $k \in \mathbb{Z}_+$, we have

$$\sup_{x \in D_m} \int_{P_x(M)} \int_0^1 |l_m'(s,\gamma)|^k ds \, \mu_x(d\gamma) \le C_1(m,k) \qquad (0.8)$$

for some positive constant $C_1(m,k)$ (which may depends on m and k).

Crucial Lemma 2: embed open subsets in compact manifolds

We construct a family of functions f_m defining a sequence of bounded connected open set $\{D_m\}_{m=1}^{\infty}$: $D_m = \{x \in M; f_m(x) > 0\}$. Then there exists a compact Riemannian manifold $\tilde{M}_m$ such that D_m is isometrically embedded into $\tilde{M}_m$ as an open set (i.e. we could view $D_m \subset \tilde{M}_m$ as an open subset of $\tilde{M}_m$).

Given a compact subset $K \subset M$ and a constant $L > 1$ (which could be taken to be arbitrarily large), then there exist positive constants $m_0(K, L) \in \mathbb{Z}_+$, $t_0(K, L)$ and $C_1(K, L)$ such that $K \subset B_o(2m_0 - 2) \subset D_{m_0} \subset \tilde{M}_{m_0}$, and for all $t \in (0, t_0]$,

$$
\begin{aligned}
|p(t,x,y) - p_{\tilde{M}_{m_0}}(t,x,y)| &\le C_1 e^{-\frac{L}{t}}, \\
|t\nabla_x \log p(t,x,y) - t\nabla_x \log p_{\tilde{M}_{m_0}}(t,x,y)|_{T_xM} &\le C_1 e^{-\frac{L}{t}}, \\
\left| t\nabla_x^2 \log p(t,x,y) - t\nabla_x^2 \log p_{\tilde{M}_{m_0}}(t,x,y) \right|_{T_xM \otimes T_xM} &\le C_1 e^{-\frac{L}{t}},
\end{aligned}
$$

Leading to

On complete and stochastically complete manifolds [Chen-xml-Wu 18+], Construct a cut off process:

1. Brownian bridge measure exists (defined to terminal value).
2. Existence of a O-U process on loop space
3. Local integration by part formula on path and loop spaces: For every F in the domain of a specific local Dirichlet form, and $h \in H_0$, we have

$$\mathbf{E}_{o,o}\left[dF\left(U.h(\cdot)\right)\right] = \mathbf{E}_{o,o}\left[F(\gamma)\left(\int_0^1 \left\langle h'(t) + \frac{1}{2}\operatorname{Ric}_{U_t} h(t) - \left(\nabla^2 \log p\left(1-t, \gamma(t), o\right)\right)\right.\right.\right.$$

4. local log sobloev.

Higher order Feynman-Kac formulas

The Feynman-Kac Formula and Brownian Bridges

Path Integration formula [Kac, Feynman, Simon]:

$$P_t^V f(x) = \mathbf{E}\left(f(x_t)e^{\int_0^t V(x_s)d_s}\right).$$

If $p(t,x,y)$ is the kernel for $\mathcal{L}$, then

$$p^V(t,x,y) = p(t,x,y)\mathbf{E}\left(e^{\int_0^t V(y_s)ds}\right)$$

where y_t is the Brownian bridge ($\mathcal{L}$-bridge) from x, ending at y at time t .
What happens if we were to differentiate the equation w.r.t. to the initial data? The Brownian bridge y_s solves:

$$dy_t = X(y_t) \circ dB_t + X_0(y_t)dt + \nabla \log p(T-t, y_t, y)\, dt.$$

The derivative flow solves:

$$dV_t = \nabla X(V_t) \circ dB_t + \nabla X_0(V_t)dt + \nabla^2 \log p(T-t, y_t, y)(V_t, \cdot\, dt.$$

1st Order Feynman-Kac Formula

Lemma [xml-Thompson'16] Assume $Ricci - 2Hessh$ is bounded from below. Then, for $\mathbb{V}_T = e^{-\int_0^T V(x_s)ds}$, and $f \in L^2 \cap L^\infty$,

$$dP_T^{h,V} f(v) = \frac{1}{t}\mathbf{E}\left[\mathbb{V}_T f(x_T) \int_0^t \langle W_s(v), u_s dB_s\rangle\right] - \frac{1}{t}\mathbf{E}\left[\mathbb{V}_T f(x_T) \int_0^t \int_0^r dV(W_s(v))\, ds\, dr\right].$$

Remark. This formula could have been deduced from known extrinsic formulas in [Elworthy-xml'94], where the derivative flow was used. We avoid some unnecessary conditions.
A formula without differentiating V is also available.

Estimates

$$dP^{h,V}_T f(v) = \frac{1}{t}\mathbf{E}\left[\mathbb{V}_T f(x_T)\int_0^t \langle W_s(v), u_s dB_s\rangle\right] - \frac{1}{t}\mathbf{E}\left[\mathbb{V}_T f(x_T)\int_0^t\int_0^r dV(W_s(v))\,ds\,dr\right].$$

Lemma 2. [xml-Thompson'16]

$$|\nabla \log P^{h,V}_t f|_{x_0} \le \frac{1}{\sqrt{t}}\Big(2C_1\mathcal{H}_t(f,x_0)\Big)^{\frac{1}{2}} + t|\nabla V|_\infty C_2, \quad t>0.$$

$$\mathcal{H}_t(f,x_0) := \mathbb{E}\left[\frac{f(x_t)\mathbb{V}_t}{P^{h,V}_t f(x_0)}\log\left(\frac{f(x_t)\mathbb{V}_t}{P^{h,V}_t f(x_0)}\right)\right].$$

$$|\nabla \log p^{h,V}_t|_{x_0} \le \frac{1}{\sqrt{t}}\sqrt{2C_1}\Big(\sup_{y\in M}\log\frac{p^h_t(y,y_0)}{p^h_{2t}(x_0,y_0)} + 2t(\sup V - \inf V)\Big)^{\frac{1}{2}} + t|\nabla V|_\infty C_2.$$

Given Harnack inequality Gaussian kernels implies good estimates. P. Li, S.-T. Yau, Grigoryhan, Saloffe-Coste, Ndumn,... Bakry-Qian, Wang, Chen, Kumagai,.....

Precise Gaussian

Corollary. Assume in addition, $\Phi^h - V \le c$ and $|\nabla h| + |\nabla \log J| \le c$. Then

$$\frac{|\nabla p_T^{h,V}(\cdot, y_0)|_{x_0}}{k_T(x_0, y_0)} \le C e^{h(y_0)-h(x_0)} |\beta_T^h|_\infty \left(\frac{d(x_0, y_0)}{T} + |\nabla h|_\infty + \frac{1}{\sqrt{T}} + T|dV|_\infty \right).$$

Local estimates: very standard even for hypoelliptic operators. S.-T. Yau, P. Li, Cheng, Gromov, Taylor,...
Global estimates are used in analysis on loop spaces: Sheu, Malliavin-Stroock, ...

Doubly damped stochastic parallel transport equation

doubly damped stochastic parallel transport

We introduce a doubly damped stochastic parallel transport driven by $\mathcal{R}$ and Θ (stochastic integration)

$$\langle \Theta(w_1)w_2, w_3\rangle = \left\langle \left(\nabla_{w_3}\mathrm{Ric}^\sharp\right)(w_2) - \left(\nabla_{w_2}\mathrm{Ric}^\sharp\right)(w_3), w_1 \right\rangle - \left\langle \left(\nabla_{w_1}\mathrm{Ric}^\sharp\right)(w_2), w_3 \right\rangle .$$

$$\left(\frac{D}{dt}\right) W_t^{(2)} = \left(-\frac{1}{2}\mathrm{Ric}^\sharp + \nabla dh\right)(W_t^{(2)}) + \frac{1}{2}\Theta\left(W_t(v_2)\right) W_t(v_1) + \mathcal{R}(d\{x_s\}, W_t(v_2))W_t(v_1),$$

$$
\begin{aligned}
\mathrm{Hess}\, P_t f(v_1, v_2) =& \frac{4}{t^2}\mathbf{E}\left[f(x_t) \int_{t/2}^{t} \langle d\{x_s\}, W_s(v_1)\rangle \int_0^{t/2} \langle d\{x_s\}, W_s(v_2)\rangle \right] \\
&+ \frac{2}{t}\mathbf{E}\left[f(x_t) \int_0^{t/2} \langle d\{x_s\}, W_s^{(2)}\rangle \right].
\end{aligned}
$$

A formula in [Elworthy-xml' 94] using Derivative flow and second order derivatives, [Arnaudon-Plank-Thalmaier'03] has a local intrinsic version, another version for the Laplacian in [Elworthy-xml'98].
A formula with potential V and for manifolds with a pole can be reduced. Estimates!.

Second Order Feynman-Kac Formula

[Second Order Feynman-Kac Formula, xml17] Assume **C1**. Let V be a bounded Hölder continuous function. Then for any $f \in \mathcal{B}_b(M; \mathbf{R})$,

$$
\begin{aligned}
&\mathrm{Hess}\, P_t^{h,V} f(v_1, v_2) = e^{-V(x_0)t} \mathbf{E}\left[f(x_t) N_t\right] \\
&+ e^{-V(x_0)t} \mathbf{E}\left[f(x_t) \frac{2}{t} \int_0^{t/2} \langle d\{x_s\}, W_s^{(2)}(v_1, v_2) \rangle \right] \\
&+ e^{-V(x_0)t} \int_0^t \mathbf{E}\left[f(x_t) \frac{2\mathbb{V}_{t-r,t}}{t-r} \int_0^{(t-r)/2} \langle d\{x_s\}, W_s^{(2)}(v_1, v_2) \rangle \right] dr \\
&\quad + e^{-V(x_0)t} \int_0^t \mathbf{E}\left[f(x_t) \mathbb{V}_{t-r,t} N_{t-r}\right] dr.
\end{aligned}
$$

A local formula are obtained in Chen-xml-Wu 18+, integration by parts formula leads to an infinite dimensional process on the path space...

Open Question

Genralize this to the semi-elliptic hypoelliptic case, and /or to time dependent potential.
Example:

$$dx_t = y_t,$$
$$dy_t = -x_t dt + dB_t.$$