The lemma of de la Valleé-Poussin

27 April 2006

Here you will find a version of the classical lemma of de la Vallée-Poussin with a proof; a similar one can be found in [1].

Proposition 1. *Let μ be a positive Borel measure on $(0, +\infty)$, and $f : (0, +\infty) \to \mathbb{R}$ a nonnegative μ-integrable function. Then there is a measurable function $\Phi : [0, +\infty) \to [0, +\infty)$ which is increasing, such that $\lim_{y\to\infty} \Phi(y) = \infty$, and*

$$\int_0^\infty \Phi f \mu < +\infty.$$

In addition, the function Φ can be chosen so that it is strictly increasing, $\Phi(0) = 0$, Φ is $\mathcal{C}^\infty$, concave, and such that $\Phi(y) \leq y$ for all $y \geq 0$.

If $G : [0, +\infty) \to \mathbb{R}$ is a nonnegative function such that $\lim_{y\to\infty} G(y) = +\infty$ and, for some $\epsilon > 0$ and all $y \in [0, \epsilon]$, $G(y) \geq \epsilon y$, then Φ can be also chosen to be less than G.

Proof. Define

$$F(x) := \int_x^\infty f\mu$$

which is a decreasing function and tends to zero as $x \to \infty$ (as f is integrable). Define

$$a_n := \inf\{x > 0 \mid F(x) < 1/n^2\} \in \mathbb{R}, \quad n \geq 1,$$

and consider the increasing sequence $\{x_n\}_{n\geq 0}$ given by

$$\begin{aligned} x_0 &:= 0 \\ x_{n+1} &:= \max\{x_n + 1, a_{n+1} + 1\}. \end{aligned}$$

The point of this sequence is that $x_n \to \infty$ when $n \to \infty$ (which is not necessarily true of a_n) and that

$$F(x_n) \leq \frac{1}{n^2}.$$

Finally, we can define ϕ:

$$\chi_n := \chi_{[x_n,\infty)} \quad \text{for } n \geq 0$$
$$\phi := \sum_{n=0}^{\infty} \chi_n.$$

The function ϕ is well defined because for every $x > 0$, $\phi(x)$ is given by a finite sum. Actually, we could define ϕ equivalently as

$$\phi(x) = n + 1 \quad \text{for } x \in [x_n, x_{n+1}), \quad n \geq 0.$$

It is clear that $\lim_{x\to\infty} \phi(x) = \infty$, as $\phi(x) > n + 1$ for $x > x_n$. Also, the integral of ϕf is finite because

$$\int_0^\infty \phi f \mu = \int_0^\infty \left(\sum_{n=0}^\infty \chi_n\right) f \mu = \sum_{n=0}^\infty \int_0^\infty \chi_n f \mu = \sum_{n=0}^\infty F(x_n) \leq \sum_{n=0}^\infty \frac{1}{n^2} < +\infty.$$

(The monotone convergence theorem justifies the interchange of sums and integral here.)

Now, let us find a function Φ in these conditions, which is also concave and strictly increasing, with $\Phi(0) = 0$ and $\Phi(y) \leq y$ for $y \geq 0$. With the help of ϕ and the above sequence $\{x_n\}$, we will define Φ recursively as follows:

$$d_0 := 1;$$
$$\Phi(0) = 0;$$
$$d_{n+1} := \min\left\{d_n, \frac{n+1-\Phi(x_n)}{x_{n+1}-x_n}\right\} \quad \text{for } n \geq 0$$
$$\Phi(x) := \Phi(x_n) + d_{n+1}(x - x_n) \quad \text{for } n \geq 0, \quad x \in [x_n, x_{n+1}].$$

First, note that Φ is continuous and $\Phi(0) = 0$ by definition. Its derivative on the interval (x_n, x_{n+1}) is d_{n+1}; as $\{d_n\}$ is decreasing and positive, Φ is concave and strictly increasing, and as $d_0 = 1$, we have $\Phi(y) \leq y$ for $y \geq 0$.

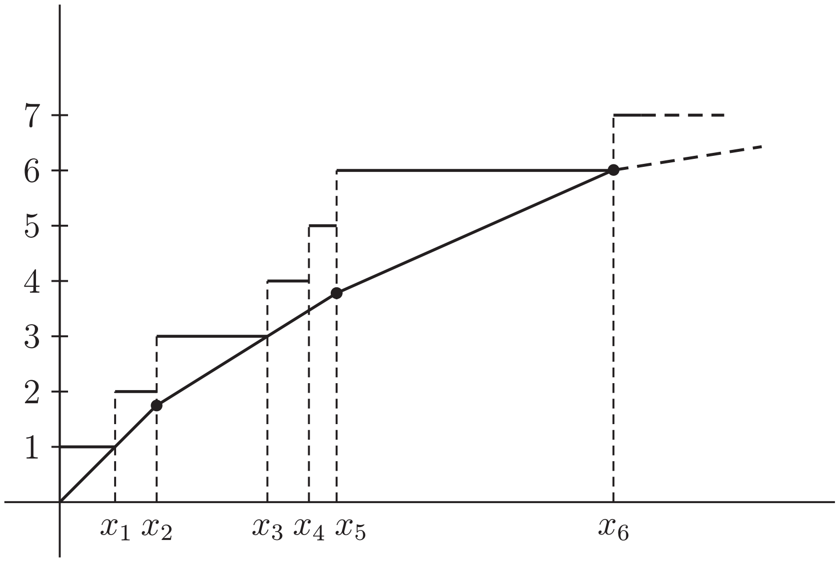

Figure 1: Definition of Φ. The step function is ϕ, and the piecewise linear one is Φ. The scales on the axes are not the same.

Also, $\Phi(x)$ is smaller than $\phi(x)$, as for x on the interval $[x_n, x_{n+1})$ ($n \geq 0$) one has

$$\begin{aligned}\Phi(x) &= \Phi(x_n) + d_{n+1}(x - x_n) \\ &\leq \Phi(x_n) + \frac{n+1-\Phi(x_n)}{x_{n+1}-x_n}(x_{n+1}-x_n) = n+1 = \phi(x).\end{aligned}$$

So the function Φf is still μ-integrable (as ϕf is). Note that the latter inequality, written for $x = x_{n+1}$, also proves that $\Phi(x_n) \leq n$ for $n \geq 0$. Also, $\lim_{x\to\infty} \Phi(x) = \infty$. To prove this, observe that d_n is always positive (as $\Phi(x_n) \leq n < n+1$), so Φ is strictly increasing. Consider the set of the n such that d_{n+1} is different from d_n; if it is finite, then from some point on Φ has a constant positive slope and hence it tends to ∞; if it is infinite, then for all such n one has

$$\begin{aligned}\Phi(x_{n+1}) &= \Phi(x_n) + d_{n+1}(x_{n+1} - x_n) \\ &= \Phi(x_n) + \frac{n+1-\Phi(x_n)}{x_{n+1}-x_n}(x_{n+1}-x_n) = n+1.\end{aligned}$$

(The equality holds because d_{n+1} is not d_n, so it must be the other quantity in the minimum). So $\lim_{x\to\infty} \Phi(x) = \infty$.

Now we can find a function Ψ with the same properties as Φ, and which is also $\mathcal{C}^\infty$: extend Φ to all of $\mathbb{R}$ as

$$\Phi(x) := d_1 x \qquad \text{for } x \leq 0.$$

Take a "bump function" $\rho : \mathbb{R} \to \mathbb{R}$ which is $\mathcal{C}^\infty$, nonnegative, with integral 1, symmetric about the $x = 0$ axis and with support contained in $[-1/2, 1/2]$. The function

$$\Psi(x) := (\Phi * \rho)(x) = \int_{-\infty}^{\infty} \Phi(x-y)\rho(y)\, dy = \int_{-\infty}^{\infty} \Phi(y)\rho(x-y)\, dy$$

is the one we are looking for: $\Psi(0) = 0$, as Φ is equal to $d_1 y$ on the interval $[-1/2, 1/2]$ (recall that $x_1 \geq 1$) and ρ is symmetric, so

$$\Psi(0) = \int_{-\infty}^{\infty} \Phi(y)\rho(-y)\, dy = d_1 \int_{-1/2}^{1/2} y\rho(-y)\, dy = 0.$$

Ψ is $\mathcal{C}^\infty$, being a regularization of Φ by a $\mathcal{C}^\infty$ function; it is less than x, as for $0 \leq x \leq 1/2$ we know that $\Psi(x) = d_1 x \leq x$, and for $x \geq 1/2$ we have, using the symmetry of ρ and the bound for Φ,

$$\begin{aligned}\Psi(x) = \int_{-\infty}^{\infty} \rho(y)\Phi(x-y)\, dy &\leq \int_{-\infty}^{\infty} \rho(y)(x-y)\, dy \\ &= x\int_{-\infty}^{\infty} \rho(y)\, dy - \int_{-\infty}^{\infty} \rho(y) y\, dy = x.\end{aligned}$$

(Note that $\Phi(x)$ is *not* less than x for $x < 0$, so this calculation does not work for $0 \leq x < 1/2$). Ψ is concave and strictly increasing because Φ is, and convolution with a positive function preserves this; $\Psi(x)$ tends to ∞ when $x \to \infty$, and if we observe that for $x \geq 0$

$$\Psi(x) = \int_{-\infty}^{\infty} \Phi(y)\rho(x-y)\, dy \leq \|\rho\|_\infty \Phi(x+1/2) \leq \|\rho\|_\infty (\Phi(x) + \Phi(1/2)), \quad (1)$$

(note that Ψ is sublinear, as it is concave and $\Psi(0) = 0$, so $\Psi(x+y) \leq \Psi(x) + \Psi(y)$ for $x, y \geq 0$), then it is clear that Ψf is integrable on $(0, +\infty)$.

Finally, let us see that Ψ can be chosen to be less than a G in the conditions of the statement. Call

$$b_n := \inf\{x \in [0, +\infty) \mid G(x) > n+1\} < +\infty.$$

In the definition at the beginning of the proof, put $y_n := \max x_n, b_n + 1$, and define ϕ using y_n instead of x_n. Then,

$$\phi(x) \leq G(x) + 1 \quad \text{for } x \geq x_1.$$

Define Φ accordingly (so $\Phi(x) \leq G(x)+1$ for $x \geq x_1$), and choose $\delta > 0$ such that

$$\delta \leq \min\{1, 1/\|\rho\|_\infty, 1/(\|\rho\|_\infty \Phi(1/2))\}.$$

Then define Ψ as the convolution above, times δ:

$$\Psi := \delta\Phi * \rho.$$

The bound in (1) proves that $\Psi(y) \leq G(y)$ for $y \geq x_1$, and this Ψ still satisfies all the other properties of the proposition. Now we only have to choose another $\delta > 0$ such that

$$\delta\Psi'(0) \leq \epsilon$$
$$\delta\Psi(x) \leq G(x) \quad \text{for } \epsilon \leq x \leq x_1,$$

and then $\delta\Psi$ is less than G (recall that $G(x) \geq \epsilon x$ for $x \in [0, \epsilon]$ and Ψ is concave) and satisfies all the other properties. □

In the rest of this section, S will be a set, $\mathcal{A}$ will be a σ-algebra of subsets of S and μ be a positive measure on $\mathcal{A}$.

Proposition 2. *Consider the positive measure space $(S, \mathcal{A}, \mu)$. If $f : S \to \mathbb{R}$ is a nonnegative μ-integrable function, then there is a continuous function $\Lambda : [0, +\infty) \to [0, +\infty)$ which is increasing, such that $\lim_{y\to\infty} \Lambda(y)/y = \infty$, and*

$$\int_0^\infty \Lambda(f(y))\mu(y) < +\infty.$$

The function Λ can be chosen so that $\Lambda(0) = 0$, Λ is $\mathcal{C}^\infty$, and strictly convex.

If $H : [0, +\infty) \to \mathbb{R}$ is an absolutely continuous function so that $G = H'$ is in the conditions of G in proposition 1, then Λ can be chosen to be less than H.

This result is a corollary of the previous proposition if one uses the concept of the *distribution function* of a given function f:

Definition 3. If $f : S \to \mathbb{R}$ is a nonnegative μ-integrable function, then its *distribution function* is the function $F_f : (0, +\infty) \to [0, +\infty)$ given by

$$F_f(\lambda) := \mu\{y \in X \mid f(y) > \lambda\} \quad \text{for } \lambda > 0.$$

Note that the set $\{y \in X \mid f(y) > \lambda\}$ is measurable, as f is. It is clear that F_f is decreasing, so in particular it is Borel measurable. The following lemma gives a way to calculate the integral of $\varphi(f)$ for suitable functions ϕ knowing only the distribution function F_f.

Lemma 4. *Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a nonnegative $\mathcal{C}^1$ function such that $\varphi(0) = 0$, and $f : S \to \mathbb{R}$ a nonnegative μ-integrable function. Then*

$$\int_S \varphi(f(x))\mu(x) = \int_0^\infty F_f(\lambda)\varphi'(\lambda)\, d\lambda.$$

Proof. To prove this, note first that the function

$$\begin{aligned} G : S \times [0, +\infty) &\to \mathbb{R} \\ (x, t) &\mapsto f(x) - t \end{aligned}$$

is measurable for the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$, as it is a sum of two measurable functions. Hence, the set $\{(x, t) \in S \times [0, +\infty) \mid f(x) < t\}$ is measurable, and therefore the function

$$\begin{aligned} \chi : S \times [0, +\infty) &\to \mathbb{R} \\ (x, t) &\mapsto \begin{cases} 1 & \text{if } f(x) < t \\ 0 & \text{if } f(x) \geq t \end{cases} \end{aligned}$$

is measurable. Observe that

$$F_f(\lambda) = \int_S \chi(x, \lambda)\mu(x) \quad \text{for } \lambda > 0.$$

Hence we can apply Fubini's theorem and write

$$\int_0^\infty F_f(\lambda)\varphi'(\lambda)\, d\lambda = \int_0^\infty \int_S \chi(x, \lambda)\mu(x)\varphi'(\lambda)\, d\lambda$$
$$= \int_S \int_0^\infty \chi(x, \lambda)\varphi'(\lambda)\, d\lambda\mu(x) = \int_S \int_0^{f(x)} \varphi'(\lambda)\, d\lambda\mu(x) = \int_S \varphi(f(x))\mu(x).$$

This proves the lemma. ∎

Now we can prove proposition 2:

Proof of proposition 2. The previous lemma proves that $\int_S f\mu = \int_0^\infty F_f(\lambda)\,d\lambda$, so F_f is integrable. Proposition 1 then shows that there is a $\mathcal{C}^\infty$ nonnegative concave function on $[0,+\infty)$, which we call Λ', such that $\Lambda'(0) = 0$, $\lim_{\lambda\to\infty} \Lambda'(\lambda) = +\infty$ and

$$\int_0^\infty F_f(\lambda)\Lambda'(\lambda)\,d\lambda < +\infty.$$

We define Λ as its primitive:

$$\Lambda(\lambda) := \int_0^\lambda \Lambda'(y)\,dy.$$

Then Λ clearly fulfills the requirements of the proposition; in particular,

$$\int_S \Lambda(f(x))\mu(x) = \int_0^\infty F_f(\lambda)\Lambda'(\lambda)\,d\lambda < +\infty,$$

and also, using l'Hôpital's rule,

$$\lim_{\lambda\to\infty} \Lambda(\lambda)/\lambda = \lim_{\lambda\to\infty} \Lambda'(\lambda) = +\infty.$$

Finally, if H is in the conditions of the proposition, we may choose Λ' less than H' and the result follows. □

1 About this text

This document has been written by José Alfredo Cañizo. For comments or suggestions write to `ozarfreo@yahoo.com`. The latest version should be at `http://www.ugr.es/~ozarfreo/tex` .

References

[1] C. Dellacherie and P. A. Meyer. *Probabilités et potentiel*, chapter I-IV, pages 85–115. Hermann, Paris, 1975.