MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Course notes - Handout 1

José A. Cañizo

September 28, 2012

1.1 Basics of Set Theory

1.1.1 Sets

In order to start from zero we would need to talk about the foundations of mathematics, which is usually based on Set Theory: once we have properly defined sets, everything else can be defined in terms of them. A fully rigorous definition of sets can be given through the axioms of Set Theory, which we will not go into in this course. If you want to know more about that you can search for information on Zermelo-Fraenkel set theory, which is the foundation we will be implicitly assuming. So, for our purposes we will be happy with the following informal definition:

Definition 1.1 (Set)**.** A *set* is a collection of objects.

We will usually denote the elements of a set by lowercase letters: $a, b, c...$ and sets by uppercase letters: $A, B, C...$ We will often consider sets of sets, and we denote them by script letters: $\mathscr{A}, \mathscr{B}, \mathscr{C}...$

If an element a is in a set A then we denote this symbolically as $a \in A$, read in many different ways: "a belongs to A" or "a is an element of A" or "a is in A" or "a is a member of A". We write that a is *not* an element of A by $a \notin A$.

Definition 1.2. Let A and B be sets. We say that A *is contained in* B or A is a *subset* of B, and denote it by $A \subseteq B$, when every element of A is also an element of B.

Likewise, we say that A *contains* B, and denote it by $A \supseteq B$, when every element of B is also an element of A (which is the same as $B \subseteq A$.)

Two sets A and B are *equal* when $A \subseteq B$ and $B \subseteq A$. We denote this by $A = B$.

(This may seem a bit pedantic, but is is useful to have a clear idea of what these terms mean.)

Ways to specify a set There are two ways to specify a particular set: one of them is by giving explicitly a list of its elements wrapped in curly brackets, such as for example

$$A = \{1, 2, 3, 4, 5\}, \quad B = \{1, 3, 5\}, \tag{1}$$

so that A is the set of numbers from 1 to 5 and B is the set consisting of the numbers 1, 3 and 5. The elements of a set could be anything. We may write

$$C = \{\clubsuit, \diamondsuit, \spadesuit, \heartsuit\}, \tag{2}$$

to specify a set whose elements are $\clubsuit$, $\diamondsuit$, $\spadesuit$ and $\heartsuit$. Then it is true that

$$2 \in A, \quad 7 \notin B, \quad \diamondsuit \in C.$$

Notice that Definition 1.2 implies that the order in which we write the elements in curly brackets, or whether we repeat them, does not matter. For example,

$$\{1, 2, 3\} = \{1, 3, 2\} = \{3, 3, 2, 1, 1\}.$$

Another way to specify a set one is by specifying it as the set of all elements which satisfy a certain property. For example

$$B = \{i \in A \mid i \text{ is odd.}\},$$

which defines B to be the same as above, assuming A is still the set given by (1). Or

$$D = \{x \in C \mid x \text{ is colored black}\},$$

which would make

$$D = \{\clubsuit, \spadesuit\},$$

assuming that we have defined what "colored black" means for the elements of C. The part before the separator "|" is usually used to specify that the set we are defining is a subset of some other set, C in the last example. The part after the separator "|" is reserved for specifying more complicated properties.

The empty set There is a special set, called the *empty set*, which has no elements. It is usually denoted by $\emptyset$. Notice that, due to Definition 1.2, for any set X it is always true that

$$\emptyset \subseteq X$$

no matter what X is. This is just because "every" element of $\emptyset$ is an element of X, since there are none to contradict this.

Complements

Definition 1.3 (Complement)**.** Let A, X be sets such that $A \subseteq X$. The *complement* of A in X, denoted $X \setminus A$, is the set of all elements of X which are not elements of A; this is,

$$X \setminus A := \{x \in X \mid x \notin A\}.$$

Sets of sets Once we have defined the sets A, B and C as in (1) and (2), the following is also a perfectly good set:

$$\mathscr{A} = \{A, B, C\}.$$

It is the set whose elements are A, B and C, which happen to be sets themselves. Here it is true that

$$A \in \mathscr{A}, \quad C \in \mathscr{A}, \quad \clubsuit \notin \mathscr{A}, \quad \clubsuit \in A.$$

We use a script letter for $\mathscr{A}$ just to remind ourselves that its elements are sets; otherwise, $\mathscr{A}$ is just a set like any other.

The power set Given a set X we define

$$\mathscr{P}(X) := \{A \text{ set } \mid A \subseteq X\}.$$

This is called the *power set* of X; it is the set of all possible subsets of X. For example,

$$\mathscr{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

In particular it is always true that

$$X \in \mathscr{P}(X) \quad \text{and} \quad \emptyset \in \mathscr{P}(X).$$

Indexed collections of sets A collection of sets indexed in a set Y is just a collection of sets called A_α (or similar), where α can be any element of Y. For example, if we have three sets called A_1, A_2, A_3, that's a collection indexed in the set $\{1, 2, 3\}$. If you have four sets called $X_\clubsuit$, $X_\diamondsuit$, $X_\spadesuit$, $X_\heartsuit$, that's a collection indexed in the set $\{\clubsuit, \diamondsuit, \spadesuit, \heartsuit\}$. Similarly, if for each real number r we have somehow defined a set called B_r, all those form a collection of sets indexed in $\mathbb{R}$. We often denote a collection of sets indexed in a set Y by $\{A_\alpha \mid \alpha \in Y\}$.

1.2 Union and intersection

Definition 1.4 (Union, intersection of two sets)**.** Let A and B be sets.

We define the *union* $A \cup B$ as the set

$$A \cup B := \{x \mid x \in A \text{ or } x \in B\}.$$

This is: $A \cup B$ is the set whose elements are all the elements of A, plus all the elements of B.

We define the *intersection* $A \cap B$ as the set

$$A \cap B := \{x \mid x \in A \text{ and } x \in B\}.$$

This is: $A \cap B$ is the set of all elements which are both in A and B.

Remark 1.5. Actually, the existence of the union of two sets is usually an *axiom* in Set Theory.

For example, we have

$$\{1,2,3\} \cup \{4,5,6\} = \{1,2,3,4,5,6\}$$
$$\{1,2\} \cup \{2,3\} = \{1,2,3\}$$
$$\{7\} \cup \{\clubsuit\} = \{7,\clubsuit\}$$

and

$$\{1,2,3\} \cap \{4,5,6\} = \emptyset$$
$$\{1,2\} \cap \{2,3\} = \{2\}.$$

In much the same way one can define the union or intersection of any collection of sets.

Definition 1.6 (Union, intersection of a collection of sets)**.** Let X be a set, and $\mathscr{A} \subseteq \mathscr{P}(X)$ a collection of subsets of X.

We define the *union* of all $A \in \mathscr{A}$ as the set

$$\bigcup_{A \in \mathscr{A}} A := \{x \in X \mid x \in A \text{ for some } A \in \mathscr{A}\}.$$

This is: it is the set of all the elements which are in *some* set $A \in \mathscr{A}$.

We define the *intersection* of all $A \in \mathscr{A}$ as the set

$$\bigcap_{A \in \mathscr{A}} A := \{x \in X \mid x \in A \text{ for all } A \in \mathscr{A}\}.$$

This is: it is the set of elements common to *all* of the sets $A \in \mathscr{A}$.

1.3 Cartesian products and functions

You probably know what the cartesian product of two sets is. We can define it as follows:

Definition 1.7. Given two sets A and B, their *cartesian product* $A \times B$ is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. This is:

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$

We remark that we are assuming the meaning of "ordered pair" here. In order to make this fully rigorous one would have to go into further details of Set Theory, and since we don't want to do this now, we will leave it this way. It is important to notice that in ordered pairs, the order does matter: $(1, 2) \neq (2, 1)$. The a and b in a pair (a, b) are called its *coordinates* or sometimes *entries*.

You also probably know what a function is. However, in order to make the concept more rigorous, their definition may seem strange at first sight:

Definition 1.8. Let A, B be sets. A *function* from A to B is a subset of $f \subseteq A \times B$ such that for each $a \in A$ there exists a unique pair in f whose first coordinate is a.

This is really the same as the more usual idea you may have of a function: a way to assign to each element of A a unique element of B.

We will also consider later the cartesian product of any collection of sets $\{A_\alpha \mid \alpha \in Y\}$ indexed in an arbitrary set Y. It is denoted by $\prod_{\alpha \in Y} A_\alpha$ and informally, it is just the set of all "ordered" lists $(a_\alpha)_{\alpha \in Y}$ with $a_\alpha \in$, meaning that for each $\alpha \in Y$ you have one $a_\alpha \in A_\alpha$. We write this as

$$\{A_\alpha \mid \alpha \in Y\} = \{(a_\alpha)_{\alpha \in Y} \mid a_\alpha \in A_\alpha\}.$$

This is enough for what we will do later. However, it is again not fully rigorous and if you want to know one can make it more so by the following definition:

Definition 1.9. Let $\{A_\alpha \mid \alpha \in Y\}$ be a collection of sets indexed in a set Y. Their cartesian product $\prod_{\alpha \in Y} A_\alpha$ is the set of all functions

$$f : Y \to \bigcup_{\alpha \in Y} A_\alpha$$

such that $f(\alpha) \in A_\alpha$ for all $\alpha \in Y$.

This is the same thing as before, just with each entry called $f(\alpha)$, and it is defined completely in terms of previously defined concepts.

1.4 A reminder on metric spaces

Definition 1.10. Let X be a set. A *distance* on X is a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$ for all $x, y \in X$.
2. $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) = d(y, x)$ for all $x, y \in X$.
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

1.5 Topology

Definition 1.11 (Topology)**.** Let X be a set. A *topology* on X is a family $\mathscr{T} \subseteq \mathscr{P}(X)$ of subsets of X satisfying the following properties:

1. $\emptyset \in \mathscr{T}$, $X \in \mathscr{T}$.
2. If $\mathscr{A} \subseteq \mathscr{T}$ is any subfamily of $\mathscr{T}$ then

$$\bigcup_{A \in \mathscr{A}} A \in \mathscr{T}. \tag{3}$$

3. If $\mathscr{A} \subseteq \mathscr{T}$ is any *finite* subfamily of $\mathscr{T}$ then

$$\bigcap_{A \in \mathscr{A}} A \in \mathscr{T}. \tag{4}$$

We refer to the pair $(X, \mathscr{T})$ as a *topological space.*

We also define now several concepts related to this:

Definition 1.12. Let $(X, \mathscr{T})$ be a topological space.

1. The elements of X are called the *points* of X.
2. The elements of $\mathscr{T}$ are called the *open sets* of X, so that U is open in X if and only if $U \in \mathscr{T}$.
3. A subset C of X is said to be *closed* if its complement $X \setminus C$ is open (this is, if $X \setminus C \in \mathscr{T}$.)
4. The set A is said to be a *neighbourhood* of the point x if and only if there is an open set $U \in \mathscr{T}$ such that $x \in U \subseteq A$.
5. If $\{x\}$ is an open set, then x is said to be an *isolated* point of X.

1.5.1 Some examples

The usual topology of $\mathbb{R}^d$ These "open sets" are a generalization of the open sets of the d-dimensional euclidean space $\mathbb{R}^d$, which you may be more familiar with:

Definition 1.13 (Open and closed sets of $\mathbb{R}^d$)**.** A subset $U \subseteq \mathbb{R}^d$ is called *open* if for every $x \in U$ there exists $r > 0$ such that $B(x, r) \subseteq U$. A subset $U \subseteq \mathbb{R}^d$ is called *closed* if its complement $\mathbb{R}^d \setminus U$ is open.

One of the main examples of a topology on $\mathbb{R}^d$ is the collection of all open sets:

$$\mathscr{T} := \{U \subseteq \mathbb{R}^d \mid U \text{ open}\}.$$

This is called *the usual topology* of $\mathbb{R}^d$.

Exercise 1.14. *Prove that $\mathscr{T}$ is a topology on $\mathbb{R}^d$.*

The trivial topology If X is any set, then

$$\mathscr{T} := \{\emptyset, X\}$$

is called the *trivial topology* in X. You can check (rather trivially) that it is indeed a topology.

The discrete topology Another trivial example of a topology in a set X is $\mathscr{P}(X)$, which is called the *discrete topology* in X. This is not too interesting for opposite reasons as the trivial topology: it has too many open sets instead of too few. It is called "discrete" because you may imagine a space like this as consisting of many unrelated points with no relation to one another whatsoever.

Exercise 1.15. *Prove that the discrete topology is indeed a topology.*

The cofinite topology In any set X the following is called the *cofinite topology*:

$$\mathscr{T} := \emptyset \cup \{A \subseteq X \mid X \setminus A \text{ is finite.}\}$$

Exercise 1.16. *Prove that it is a topology.*

The cocountable topology In a similar way the following is called the *cocountable topology*:

$$\mathscr{T} := \emptyset \cup \{A \subseteq X \mid X \setminus A \text{ is countable.}\}$$

Exercise 1.17. *Prove that it is a topology.*

The Zariski topology on $\mathbb{Z}$ There a topology on the set of integers $\mathbb{Z}$ which has interest in algebraic geometry:

$$\mathscr{T} := \emptyset \cup \{A \subseteq \mathbb{Z} \mid \mathbb{Z} \setminus A \text{ is a finite set of prime numbers.}\}$$

This is called the *Zariski topology.*

Metric space topology If X is a metric space with distance d, then one defines an open set in much the same way as in $\mathbb{R}^d$:

Definition 1.18. Let (X, d) be a metric space. A subset $U \subseteq X$ is called *open* if for every $x \in U$ there exists $r > 0$ such that $B(x, r) \subseteq U$. A subset $U \subseteq X$ is called *closed* if its complement $X \setminus U$ is open.

Of course, $B(x, r)$ is defined in terms of the metric d:

$$B(x, r) = \{y \in X \mid d(x, y) < r\}.$$

Then the *metric topology* of X, or the topology *associated to the distance* d is the topology formed by all open sets (with the previous definition).