MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Course notes - Handout 10

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10.1 Connectedness

We say that a topological space is connected when it cannot be broken into two open pieces in a nontrivial way:

Definition 10.1. We say that a topological space X is *connected* when there do not exist two nonempty disjoint open sets U, V such that $X = U \cup V$.

Observe that the open sets U, V in the above definition have to be nonempty (otherwise we're not really breaking X in any meaningful way) and disjoint (we're not patching the space, we need to split it).

Example 10.2. *One can easily give a number of fairly trivial examples:*

1. *A set with one element (with the only possibly topology on it) is connected.*

2. *Any set with the trivial topology is connected.*

3. *A set with more than one element and the discrete topology is not connected.*

4. *The space* $[1, 2] \cup [3, 4]$*, with the topology induced from* $\mathbb{R}$*, is not connected: it is obviously split into the sets* $[1, 2]$ *and* $[3, 4]$*, both of which are open in the induced topology.*

A less trivial example is the following:

Lemma 10.3. *The interval* $[0, 1]$ *is connected.*

Proof. Assume $[0, 1]$ is not connected. Then there exists a splitting $[0, 1] = (U \cap [0, 1]) \cup (V \cap [0, 1])$ with U, V open in $\mathbb{R}$ and

$$U \cap [0, 1] \neq \emptyset, \quad V \cap [0, 1] \neq \emptyset, \quad U \cap V \cap [0, 1] = \emptyset.$$

Assume that the point 1 is in V (if not, just interchange the names of U and V). Since V is open this implies that

$$(1-\epsilon, 1] \subseteq V \quad \text{for some } \epsilon > 0. \tag{1}$$

Call $x := \sup(U \cap [0,1])$ (which is well defined since $U \cap [0,1]$ is nonempty and bounded). Then $x \neq 0$, for otherwise we have $U \cap [0,1] = \{0\}$, which is not an open set of $[0,1]$. Due to this and (1), $x \in (0, 1-\epsilon]$. Now, x must be in either U or V, but

1. If $x \in U$ then there is some interval $(x-\delta, x+\delta)$ which is in $U \cap [0,1]$, contradicting the definition of x.

2. If $x \in V$ then there is some interval $(x-\delta, x+\delta)$ which is in $V \cap [0,1]$ (and hence not in $U \cap [0,1]$), again contradicting the definition of x.

This gives a contradiction, proving that $[0,1]$ is connected. □

Corollary 10.4. *Any interval of $\mathbb{R}$ is connected. In particular, $\mathbb{R}$ is connected.*

Proof. First, note that any closed interval $[a,b]$ with $a < b$ is connected, since it is homeomorphic to $[0,1]$.

Let $I \subseteq \mathbb{R}$ be some interval, and assume it is not connected. Take U, V an open nontrivial splitting of I, and pick $a \in U$, $b \in V$. We may assume $a < b$ (if not, change their names,) so $[a,b] \subseteq I$. Then $[a,b] = (U \cap [a,b]) \cup (V \cap [a,b])$ is an open nontrivial splitting of $[a,b]$, which gives a contradiction. □

Theorem 10.5. *The image of a connected topological space by a continuous function is connected.*

Proof. Take X, Y topological spaces with X connected, and $f : X \to Y$ continuous. Assume $f(Y)$ is not connected; this is, there exist U, V nonempty disjoint open sets of $f(Y)$ such that $f(Y) = U \cup V$. The restriction $\tilde{f} : X \to f(Y)$ (obtained by reducing the image of f) is also a continuous function, so $\tilde{f}^{-1}(U)$ and $\tilde{f}^{-1}(V)$ are nonempty disjoint open sets of X, such that $X = \tilde{f}^{-1}(U) \cup \tilde{f}^{-1}(V)$. This is a contradiction, since X is connected. □

Theorem 10.6 (Intermediate value theorem)**.** *Let X be connected, and $f : X \to \mathbb{R}$ a continuous function. If f takes the values a and b and certain points, then it must take all values c in (a,b) at some (other) points.*

Proof. Assume $x, y \in X$ are such that $f(x) = a$, $f(y) = b$, with $a < b$ (otherwise the statement is empty.) Take $c \in (a,b)$. Reasoning by contradiction, if f never takes the value c then

$$f(X) = (f(X) \cap (-\infty, c)) \cup (f(X) \cap (c, +\infty))$$

is an open nontrivial splitting of $f(X)$, contradicting the fact that $f(X)$ must be connected. □

10.2 Path connectedness

Definition 10.7. A *path* on a topological space X joining two points $x, y \in X$ is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Definition 10.8. We say that a topological space X is *path-connected* when for any two points $x, y \in X$ there exists a path in X joining x and y.

Theorem 10.9. *A path-connected topological space is connected.*

Sketch of proof. If $X = U \cup V$ is a nontrivial open splitting of X, then take $x \in U$ and $y \in V$. Now if f is a path joining x and y, then $(f([0, 1]) \cap U) \cup (f([0, 1]) \cap V))$ is a nontrivial open splitting of $f([0, 1])$. This is a contradiction, since Lemma 10.3 and Theorem 10.5 show that $f([0, 1])$ is connected. □

Exercise 10.10. *Prove or disprove the following statements about subsets of* $\mathbb{R}$ *with the usual topology:*

1. $(0, 1)$ *is homeomorphic to* $[0, 1]$.
2. $(0, 1]$ *is homeomorphic to* $[0, 1)$.
3. $(0, 1]$ *is homeomorphic to* $[0, 1]$.
4. $(0, 1)$ *is homeomorphic to* $[0, 1)$.

Path-connectedness allows us to show easily that many spaces are connected. For example, a convex subset of $\mathbb{R}^d$ must be connected, because it is path-connected by definition. In particular, open and closed balls in $\mathbb{R}^d$ with any of the p-distances (for $1 \leq p \leq \infty$) are connected.

Exercise 10.11. *Consider the set*

$$A := \left\{ \left(x, \sin \frac{1}{x}\right) \mid x \in (0, 1) \right\}.$$

1. ***(1 mark)*** *Describe explicitly the closure* $\overline{A}$ *of* A*, and give a justification for the answer.*
2. ***(1 mark)*** *Show that* $\overline{A}$ *is connected but not path-connected.*