MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Course notes - Handout 2

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2.1 A reminder on metric spaces

Definition 2.1. Let X be a set. A *distance* on X is a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$ for all $x, y \in X$.
2. $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) = d(y, x)$ for all $x, y \in X$.
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A *metric space* is a set X together with a distance d on X.

Example 2.2. *In $\mathbb{R}^d$, the p-distance (for $p \geq 1$) is*

$$d_p(x, y) = \left(|x_1 - y_1|^p + \dots + |x_d - y_d|^p\right)^{1/p} \quad \text{for } x, y \in \mathbb{R}^d,$$

where $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$. We will not prove here that it is indeed a distance, but we notice that $p \geq 1$ is needed in order to ensure that point 4 of Definition 2.1 is satisfied. One also defines the ∞-distance as

$$d_\infty(x, y) = \max\left\{|x_1 - y_1|, \dots, |x_d - y_d|\right\} \quad \text{for } x, y \in \mathbb{R}^d.$$

Exercise 2.3. *Prove that d_p, as defined in the previous example, is a distance on $\mathbb{R}^d$.*

Definition 2.4. Let X be a set and d_1, d_2 two distances on X. We say that d_1 and d_2 are *equivalent* when there exist numbers $C_1, C_2 > 0$ such that

$$C_1 d_1(x, y) \leq d_2(x, y) \leq C_2 d_1(x, y) \quad \text{for all } x, y \in X.$$

Exercise 2.5. *Obviously, a distance is always equivalent to itself. Prove that also:*

1. *If d_1 and d_2 are equivalent distances, then d_2 and d_1 are also.*
2. *If d_1 and d_2 are equivalent distances, and d_2 and d_3 are equivalent distances, then d_1 and d_3 are also.*

This is: "equivalence", as we have defined it, is an equivalence relation. This justifies the name we have given to it, since otherwise it would be misleading.

Exercise 2.6. *Prove that all p-distances on $\mathbb{R}^d$, for $p \in [1, \infty]$, are equivalent.*

Metric space topology If X is a metric space with distance d, then one defines an open set in much the same way as in $\mathbb{R}^d$:

Definition 2.7. Let (X, d) be a metric space. A subset $U \subseteq X$ is called *open* if for every $x \in U$ there exists $r > 0$ such that $B(x, r) \subseteq U$. A subset $U \subseteq X$ is called *closed* if its complement $X \setminus U$ is open.

Of course, $B(x, r)$ is defined in terms of the metric d:

$$B(x, r) = \{y \in X \mid d(x, y) < r\}.$$

Then the *metric topology* of X, or the topology *associated to the distance d* is the topology formed by all open sets (with the previous definition).

Proposition 2.8. *Let X be a set and d_1, d_2 two equivalent distances on X. Then the topologies associated to the distance d_1 and the distance d_2 are the same.*

2.2 Continuous functions

Definition 2.9 (Continuous function)**.** Let $(X, \mathscr{T})$ and $(Y, \mathcal{S})$ be topological spaces. A map $f : X \to Y$ is said to be *continuous* if and only if $f^{-1}(U) \in \mathscr{T}$ whenever $U \in \mathcal{S}$.

This is: a map $f : X \to Y$ is continuous if and only if the inverse image of every open set is open. Note that for $f : X \to Y$ the *inverse image* of a set $A \subseteq Y$ is defined as

$$f^{-1}(A) := \{x \in X \mid f(x) \in A.\}$$

You already knew a definition of continuity for real functions, which we recall:

Definition 2.10 (Continuous function on $\mathbb{R}^d$)**.** A function $f : \mathbb{R}^d \to \mathbb{R}^d$ is said to be *continuous at a point* $x \in \mathbb{R}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that, for all $y \in \mathbb{R}^d$,

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon.$$

The function f is said to be *continuous* if it is continuous at every point $x \in \mathbb{R}^d$.

Proposition 2.11. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous with the usual topology of $\mathbb{R}^d$ (i.e. according to Definition 2.9) if and only if it is continuous at every point in the usual meaning for real functions (i.e. according to Definition 2.10.)*