MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Course notes - Handout 3

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3.1 The induced topology

Any subset A of a topological space "inherits" a topology from it in a natural way:

Definition 3.1 (Induced topology)**.** Let $(X, \mathscr{T})$ be a topological space, and $A \subseteq X$ any subset. Then we define a topology $\mathscr{T}_A$ on A as follows, and called it the *induced topology* on A:

$$\mathscr{T}_A := \{U \cap A \mid U \in \mathscr{T}\}.$$

Exercise 3.2. *Show that $\mathscr{T}_A$ is indeed a topology on A.*

3.2 Homeomorphisms

As with any mathematical structure you have encountered, there is a notion of *equivalence* associated with the structure of a topology:

Definition 3.3 (Homeomorphism)**.** Let $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ be two topological spaces. A *homeomorphism* between X and Y is a continuous bijection $f : X \to Y$ such that its inverse f^{-1} is also continuous.

3.3 Interior and closure

We recall some important concepts from Handout 1:

Definition 3.4. Let $(X, \mathscr{T})$ be a topological space.

1. The elements of X are called the *points* of X.
2. The elements of $\mathscr{T}$ are called the *open sets* of X, so that U is open in X if and only if $U \in \mathscr{T}$.

3. A subset C of X is said to be *closed* if its complement $X \setminus C$ is open (this is, if $X \setminus C \in \mathscr{T}$.)

4. The set A is said to be a *neighbourhood* of the point x if and only if there is an open set $U \in \mathscr{T}$ such that $x \in U \subseteq A$.

5. If $\{x\}$ is an open set, then x is said to be an *isolated* point of X.

Definition 3.5. Let A be a subset of the topological space X.

1. The point x is said to be a *limit point of the set* A if and only if for every open U containing x there is some $y \in U \cap A$ such that $y \neq x$.

2. The *interior* of A, denoted $\mathrm{int}(A)$ or A°, is the union of all open subsets contained in A.

3. The *closure* of A, denoted $\mathrm{cl}(A)$ or $\overline{A}$, is the intersection of all closed sets containing A.

4. The *boundary* of A, denoted ∂A, is defined by $\partial A := \overline{A} \setminus A^\circ$.

3.4 Ways to describe a topology: bases and subbases

Definition 3.6. Let X be a space with topology $\mathscr{T}$. A collection $\mathscr{B}$ of subsets of X is said to be a *base* for $\mathscr{T}$ if and only if

1. each element of $\mathscr{B}$ is open in X (i.e. $\mathscr{B} \subseteq \mathscr{T}$) and

2. every open set U is a union of elements of $\mathscr{B}$.

A collection $\mathcal{S}$ of subsets of X is said to be a *subbase* for $\mathscr{T}$ if and only if

1. each element of $\mathcal{S}$ is open in X (i.e. $\mathcal{S} \subseteq \mathscr{T}$) and

2. every open set U is a union of *finite intersections* of elements of $\mathcal{S}$.