MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Course notes - Handout 4

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4.1 Ways to describe a topology: bases and subbases

Definition 4.1 (Base and subbase)**.** Let X be a space with topology $\mathscr{T}$. A collection $\mathscr{B}$ of subsets of X is said to be a *base* for $\mathscr{T}$ if and only if

1. each element of $\mathscr{B}$ is open in X (i.e. $\mathscr{B} \subseteq \mathscr{T}$) and
2. every open set U is a union of elements of $\mathscr{B}$.

A collection $\mathcal{S}$ of subsets of X is said to be a *subbase* for $\mathscr{T}$ if and only if

1. each element of $\mathcal{S}$ is open in X (i.e. $\mathcal{S} \subseteq \mathscr{T}$) and
2. every open set U is a union of *finite intersections* of elements of $\mathcal{S}$.

Example 4.2. *1. The collection*

$$\mathscr{B} = \{(a, b) \mid a < b \in \mathbb{R}\}$$

is a base for the usual topology on $\mathbb{R}$.

2. The collection

$$\mathscr{S} = \{(a, +\infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$$

is a subbase for the usual topology on $\mathbb{R}$.

3. Let X *be a metric space. The collection*

$$\mathscr{B} = \{B(x, r) \mid x \in X,\ r > 0\}$$

is a base for the metric topology on X.

There are many topological properties that can be checked only on the elements of a base. For example:

Exercise 4.3. *Let X, Y be two topological spaces, and let $\mathscr{B}_Y$ be a base for the topology of Y. A function $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open for all $b \in B_Y$.*

Lemma 4.4 (Which collections of subsets are a base for some topology?)**.** *Consider a set X and a collection $\mathscr{B} \subseteq \mathscr{P}(X)$ of subsets of X. Then $\mathscr{B}$ is a base for some topology $\mathscr{T}$ on X if and only if:*

1. *For each $x \in X$ there is at least one $B \in \mathscr{B}$ with $x \in B$.*

2. *For every two sets $B_1, B_2 \in \mathscr{B}$ and every $x \in B_1 \cap B_2$ there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

If these conditions are met then the topology $\mathscr{T}$ must be the one defined as follows: a set U is in $\mathscr{T}$ if and only if for every $x \in U$ there is $B \in \mathscr{B}$ such that $x \in B \subseteq U$.

Exercise 4.5. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{B}$ a base for $\mathscr{T}$. If A is a subset of X, prove that the collection*

$$\mathscr{B}_A = \{B \cap A \mid B \in \mathscr{B}\}$$

is a base for the induced topology $\mathscr{T}_A$ on A.

Definition 4.6. Take two topologies $\mathscr{T}_1$, $\mathscr{T}_2$ on a set X. We say that $\mathscr{T}_1$ is *finer* than $\mathscr{T}_2$ when $\mathscr{T}_2 \subseteq \mathscr{T}_1$ (this is, when $\mathscr{T}_2$ contains less open sets than $\mathscr{T}_1$.) Alternatively, one may say that $\mathscr{T}_2$ is *coarser* than $\mathscr{T}_1$.

Lemma 4.7 (Comparison of topologies through their bases)**.** *Consider a set X and bases $\mathscr{B}_1$, $\mathscr{B}_2$ for two topologies $\mathscr{T}_1$, $\mathscr{T}_2$ on X, respectively. Then $\mathscr{T}_2$ is finer than $\mathscr{T}_1$ if and only if for each $x \in X$ and each $B_1 \in \mathscr{B}_1$ with $x \in B$ there is $B_2 \in \mathscr{B}_2$ with $x \in B_2 \subseteq B_1$.*

4.2 More about the interior and closure

We gave the definition of the interior, closure and boundary of a subset of a topological space. We give now some further properties and some results that are helpful when finding the interiors and closures of particular sets.

Lemma 4.8. *Let $(X, \mathscr{T})$ be a topological space and A a subset of X. Then $x \in \overline{A}$ if and only if every open set U with $x \in U$ intersects A.*

If $\mathscr{B}$ is a base for $\mathscr{T}$, then $x \in \overline{A}$ if and only if every $B \in \mathscr{B}$ with $x \in B$ intersects A.

There is also a relationship between the closure and the concept of limit point:

Lemma 4.9. *Let $(X, \mathscr{T})$ be a topological space and A a subset of X. We have*

$$\overline{A} = A \cup A',$$

where A' is the set of limit points of A.