MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Course notes - Handout 5

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October 16, 2012

5.1 Convergence

Definition 5.1. Let $(X, \mathscr{T})$ be a topological space. A sequence $\{x_n\}_{n\geq 1}$ in X is said to *converge* to $x \in X$ if for every open set U with $x \in U$ there is a natural number N such that $x_n \in U$ for all $n \geq N$. In this case we say that x is a limit of the sequence $\{x_n\}_{n\geq 1}$.

Exercise 5.2. *If you consider the usual topology in $\mathbb{R}^d$ or in any metric space, this is equivalent to the usual meaning of convergence.*

In complete generality, this concept of convergence can be strange: unlike in a metric space, the limit of a sequence does not need to be unique!

Example 5.3. *There is a simple example of this: consider the set $X = \{1, 2, 3\}$ with the topology*

$$\mathscr{T} := \{\emptyset, \{1, 2, 3\}, \{1, 2\}, \{2\}, \{2, 3\}\}.$$

(This is a topology on X; check it!) Can you identify the limits of the sequence given by $x_n = 2$ for all $n \in \mathbb{N}$?

However, in spaces which satisfy a very reasonable additional requirement the limit is indeed unique:

Definition 5.4. We say that a topological space X is *Hausdorff* or T_2 when for any two points $x, y \in X$ there exist disjoint open sets $U, V \subseteq X$ with $x \in U$ and $y \in V$.

Proposition 5.5. *Let X be a Hausdorff topological space. Then the limit of a sequence $\{x_n\}$, if it exists, is unique.*

Proof. Assume that a sequence $\{x_n\}$ has two limits $x \neq y$. Since X is Hausdorff we can find disjoint open sets U, V with $x \in U$ and $y \in V$. But the terms x_n must be both in U and V for n large enough, which is a contradiction. □

Hausdorff spaces also satisfy other reasonable things:

Lemma 5.6. *In a Hausdorff topological space X, every set with a finite number of points is closed.*

Lemma 5.7. *A subset of a Hausdorff topological space is also Hausdorff.*

Since we have this concept of convergence in a general topological space, can continuity be characterized by it in the same way as one does in a metric space? This is: we know that in a metric space X with distance d a function is continuous at a point $x \in X$ if and only if $f(x_n) \to f(x)$ for any sequence $\{x_n\}$ with $x_n \to x$. Is this true in a topological space? The answer is *no* in general:

Lemma 5.8 (Continuity implies continuity by sequences)**.** *Let $f : X \to Y$ be a function between two topological spaces X and Y. If f is continuous, then for every sequence $\{x_n\}$ which converges to a point $x \in X$ it holds that $f(x_n) \to f(x)$.*

But in general the condition that $f(x_n) \to f(x)$ whenever $x_n \to x$ is not enough to ensure continuity. Since we know this is true in metric spaces, the following concept is useful:

Definition 5.9. We say that a topological space $(X, \mathscr{T})$ is *metrizable* when there exists a distance d on X such that the topology $\mathscr{T}_d$ associated to d is equal to $\mathscr{T}$.

Lemma 5.10. *Let $f : X \to Y$ be a function between two metrizable topological spaces X and Y. The following are equivalent:*

1. *The function f is continuous*
2. *For every sequence $\{x_n\}$ which converges to a point $x \in X$ it holds that $f(x_n) \to f(x)$.*

There are other things that can be characterized in terms of sequences, but only in metric spaces. For example:

Lemma 5.11. *Let X be a topological space, and $A \subseteq X$. If there exists a sequence $\{x_n\}$ in A converging to a point $x \in X$, then $x \in \overline{A}$. If the space X is metrizable, the converse is also true.*

Sketch of proof. To prove the first part, assume that $x \notin \overline{A}$. Then there is some closed set $C \supseteq A$ with $x \notin C$. Then $X \setminus C$ is an open set which will contradict the definition of convergence.

To prove the second part, notice that for all $n \in \mathbb{N}$, the ball $B(x, 1/n)$ must contain at least one point from A. That allows you to define a sequence which converges to x. $\square$