MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Course notes - Handout 6

José A. Cañizo

October 26, 2012

6.1 More on continuity

There are many ways of characterizing the concept of continuity:

Theorem 6.1. *Let X, Y be topological spaces and $f : X \to Y$ a function. The following are equivalent:*

1. *f is continuous.*
2. *For every subset $A \subseteq X$ it holds $f(\overline{A}) \subseteq \overline{f(A)}$.*
3. *For every closed set $C \subseteq Y$, $f^{-1}(C)$ is closed.*
4. *For each $x \in X$ and each neighbourhood V of $f(x)$ there is a neighbourhood U of X such that $f(U) \subseteq V$.*

Sketch of proof.

1. It is easy to see that 1 and 3 are equivalent by taking complements (recall that the concepts of open and closed are "symmetric") and knowing that $X \setminus f^{-1}(A) = f^{-1}(X \setminus A)$ for any set A.
2. To see that 3 implies 2 it is useful to notice that 2 is the same as
$$\overline{A} \subseteq f^{-1}(\overline{f(A)}) \tag{1}$$
for all sets A. Then one can show that $f^{-1}(\overline{f(A)})$ is closed and contains A.
3. To see that 2 implies 3, notice that if C is closed and, using (1) and calling $A := f^{-1}(C)$ we have $\overline{A} \subseteq f^{-1}\big(\overline{f(A)}\big) \subseteq f^{-1}\big(\overline{C}\big) = f^{-1}(C) = A$.
4. One can also easily show that 4 and 1 are equivalent in a similar way as in the proof that topological continuity is the usual continuity for real functions. One may use that a set is open if and only if it is a neighbourhood of all of its points (cf. problem sheet 1, ex. 3(a).)

□

Lemma 6.2. *Let X, Y and Z be topological spaces. The following are examples of continuous functions:*

Constant functions *Take any $y_0 \in Y$. The constant function $f : X \to Y$ given by $f(x) = y_0$ for all $x \in X$ is continuous.*

Inclusions *If $A \subseteq X$, the inclusion $i : A \to X$, $i(x) = x$ for all $x \in X$, is continuous.*

Compositions *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then their composition $g \circ f : X \to Z$ is continuous.*

Restrictions *If $f : X \to Y$ is continuous and $A \subseteq X$, then its restriction $f|_A : A \to X$ is continuous.*

Change of range *Let $f : X \to Y$ be continuous.*

1. *Let $Z \subseteq Y$ be a subset such that $f(X) \subseteq Z$. Then the function*

$$g : X \to Z \quad \text{given by} \quad g(x) = f(x) \quad \text{for all } x \in X$$

is continuous.

2. *Let Z be a topological space such that $Y \subseteq Z$ (and such tat the topology of Y is the one induced from Z.) Then the function*

$$g : X \to Z \quad \text{given by} \quad g(x) = f(x) \quad \text{for all } x \in X$$

is continuous.

Exercise 6.3. *Prove the previous lemma.*

Lemma 6.4 (Local formulation of continuity)**.** *Let X, Y be topological spaces and $f : X \to Y$ a function. If X can be written as the union of a collection of open sets $\{U_\alpha \mid \alpha \in I\}$ such that $f|_{U_\alpha}$ is continuous for each $\alpha \in I$, then f is continuous.*

Similarly, if X can be written as the union of a finite *collection of closed sets $\{C_i \mid i = 1, \ldots, n\}$ such that $f|_{U_i}$ is continuous for each $i = 1, \ldots, n$, then f is continuous.*

Sketch of proof. Use that, for any open sets $U \subseteq X$ and $V \subseteq Y$, $f^{-1}(V) \cap U = (f|_U)^{-1}(V)$. With this it is easy to rewrite $f^{-1}(V)$ as a union of open sets, each one the intersection with one of the U_α. A very similar proof works for the closed case when the collection of subsets is finite. □

Exercise 6.5. *Give counterexamples to show the following:*

1. *If the U_α are not all open then the above result does not hold in general.*

2. *If the family of sets C_i is not finite the above result does not hold in general.*