MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Course notes - Handout 8

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8.1 The product topology (II)

The following lemmas are useful when working with products, and in particular in proving Theorem 7.8. In this section we assume that I is some indexing family, $\{(X_\alpha, \mathscr{T}_\alpha) \mid \alpha \in I\}$ is a collection of topological spaces, and $\prod_{i\in I} X_\alpha$ is their product topological space whose topology we denote by $\mathscr{T}$.

Lemma 8.1 (Products and intersections)**.** *For each $\alpha \in I$ let A_α and B_α be subsets of X_α. We have*

$$(\prod_{\alpha\in I} A_\alpha) \cap (\prod_{\alpha\in I} B_\alpha) = \prod_{\alpha\in I}(A_\alpha \cap B_\alpha). \tag{1}$$

Lemma 8.2 (Projections are continuous)**.** *For any $\beta \in I$ the projection $\pi_\beta : \prod_{i\in I} X_\alpha \to X_\beta$ is continuous.*

Lemma 8.3. *For each $\alpha \in I$ let A_α be a subset of X_α. Then*

$$\prod_{\alpha\in I} A_\alpha = \bigcap_{\alpha\in I} \pi_\alpha^{-1}(A_\alpha).$$

Remark 8.4. In the product of just two topological spaces X_1, X_2 this lemma is very easy to see: notice that

$$\pi_1^{-1}(A_1) = A_1 \times X_2, \quad \pi_2^{-1}(A_2) = X_1 \times A_2,$$

and the lemma just says that

$$A_1 \times A_2 = (X_1 \times A_2) \cap (A_1 \times X_2).$$

Lemma 8.5 (The product of closed sets is closed)**.** *For each $\alpha \in I$ let C_α be a closed subset of X_α. Then $\prod_{\alpha\in I} C_\alpha$ is closed in $\prod_{\alpha\in I} X_\alpha$.*

Exercise 8.6. *Is the product of open sets always open?*

Lemma 8.7. *A subbase for the product topology of $\prod_{\alpha\in I} X_\alpha$ is given by*

$$\mathscr{S} := \left\{\pi_\alpha^{-1}(U_\alpha) \mid \alpha \in I,\ U_\alpha \in \mathscr{T}_\alpha\right\}.$$

8.2 The quotient topology

Another very useful construction in topology is that of a *quotient space*. One of the simplest ways to define it is the following:

Definition 8.8. A *relation of equivalence* $\mathscr{R}$ on a set X is a subset of $X \times X$ satisfying the following conditions: (we write $x \sim y$ or $x \sim_{\mathscr{R}} y$ to denote $(x, y) \in \mathscr{R}$)

1. For all $x \in X$ we have $x \sim x$.

2. For all $x, y \in X$ we have $x \sim y \Rightarrow y \sim x$.

3. For all $x, y, z \in X$ we have $(x \sim y$ and $y \sim z) \Rightarrow x \sim z$.

Given a relation of equivalence $\mathscr{R}$, the *equivalence class* of an element $x \in X$, denoted by x^*, is

$$x^* := \{y \in X \mid x \sim y\}.$$

Definition 8.9. Consider a topological space X with topology $\mathscr{T}$, and a relation of equivalence $\mathscr{R}$ on X. We denote by X^* the set of equivalence classes of X with the relation $\mathscr{R}$, and consider the map

$$\begin{aligned} q : X &\to X^* \\ x &\mapsto x^* \end{aligned}$$

which maps an element x to its equivalence class x^* under the relation $\mathscr{R}$.

The *quotient topological space* $X/\mathscr{R}$ is the set X^* with the topology

$$\mathscr{T}^* \equiv \mathscr{T}/\mathscr{R} := \{U \subseteq X^* \mid q^{-1}(U) \in \mathscr{T}\}.$$

The space may be denoted by X^* or $X/\mathscr{R}$, and the topology by $\mathscr{T}/\mathscr{R}$ or $\mathscr{T}^*$.

There is a special case of this definition:

Definition 8.10. Consider a topological space X and a subset $A \subseteq X$, and the relation of equivalence $\mathscr{R}$ on X defined by

$$x \sim y \Leftrightarrow (x, y \in A) \text{ or } x = y.$$

(This is: $\mathscr{R}$ is the relation of equivalence whose equivalence classes are A and all sets $\{x\}$ with $x \notin A$.) The *quotient topological space* X/A is defined as $X/\mathscr{R}$ with the above relation $\mathscr{R}$.

The intuitive meaning of a quotient space is that each equivalence class is *glued to form just one point.*

Exercise 8.11. *Consider the following sets X and subsets A. For each of them, can you find a common space Y which is homeomorphic to X/A?*

1. $X = [0, 1]$ *and* $A = \{0, 1\}$.
2. $X = \overline{B(0, 1)} \subseteq \mathbb{R}^2$ *and* $A = S(0, 1) = \{x \in \mathbb{R}^2 \mid |x| = 1\}$.

Exercise 8.12. *Consider the following sets X and relations $\mathscr{R}$. For each of them, can you find a common space Y which is homeomorphic to $X/\mathscr{R}$?*

1. $X = \mathbb{R}$, *and*
$$x \sim y \Leftrightarrow |x - y| = 1.$$
2. $X = S(0, 1) \subseteq \mathbb{R}^2$, *and*
$$(x_1, x_2) \sim (y_1, y_2) \Leftrightarrow x_1 = y_1.$$
3. $X = [0, 1] \times [0, 1]$ *and*
$$(x_1, x_2) \sim (y_1, y_2) \Leftrightarrow \begin{cases} (x_1, x_2) = (y_1, y_2) \\ \quad or \\ x_2 = 0, y_2 = 1 \text{ and } x_1 = 1 - x_2 \\ \quad or \\ x_2 = 1, y_2 = 0 \text{ and } x_1 = 1 - x_2. \end{cases}$$
4. $X = [0, 1] \cup [2, 3]$ *and*
$$x \sim y \Leftrightarrow \Big\{x = y \text{ or } \{x, y\} = \{1, 2\} \text{ or } \{x, y\} = \{0, 3\}\Big\}.$$