MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Solutions to problem sheet 1

José A. Cañizo

February 2013

Exercise 1.2. Let X and Y be sets and $f : X \to Y$ be a function. Let $A \subseteq X$, $\mathscr{A} \subseteq \mathscr{P}(X)$, $B \subseteq Y$, and $\mathscr{B} \subseteq \mathscr{P}(Y)$. We use the shorthand

$$\bigcup \mathscr{A} \equiv \bigcup_{A \in \mathscr{A}} A,$$

and similarly for $\mathscr{B}$.

1. Prove that $f\big(\bigcup \mathscr{A}\big) = \bigcup_{A \in \mathscr{A}} f(A)$.
2. Prove that $f\big(\bigcap \mathscr{A}\big) \subseteq \bigcap_{A \in \mathscr{A}} f(A)$. Under what conditions on f do you get equality for all collections $\mathscr{A}$? Give an example where equality does not hold.
3. Prove that $f^{-1}\big(\bigcup \mathscr{B}\big) = \bigcup_{B \in \mathscr{B}} f^{-1}(B)$.
4. Prove that $f^{-1}\big(\bigcap \mathscr{B}\big) = \bigcap_{B \in \mathscr{B}} f^{-1}(B)$.
5. Prove that $A \subseteq f^{-1}\big(f(A)\big)$. Under what conditions do you get equality for all sets $A \subseteq X$? Give an example where equality does not hold.
6. Prove that $f\big(f^{-1}(B)\big) \subseteq B$. Under what conditions do you get equality for all sets $B \subseteq Y$? Give an example where equality does not hold.
7. Prove that $X - f^{-1}(B) = f^{-1}(Y - B)$. What can you say about $Y - f(A)$ and $f(X - A)$? Under what conditions would these two sets be equal?

Solution. 1. If y is in $f(\bigcup \mathscr{A})$ then there exist $B \in \mathscr{A}$ and $x \in B$ with $y = f(x)$. Hence $y \in f(B)$, and in particular $y \in \bigcup_{A \in A} f(A)$. Conversely, if $y \in \bigcup_{A \in A} f(A)$, then $y \in f(B)$ for some $B \in \mathscr{A}$. In particular, $y \in f(\bigcup \mathscr{A})$, since $B \subseteq \bigcup \mathscr{A}$.

2. Take $y \in f(\bigcap \mathscr{A})$. Then $y = f(x)$ for some $x \in \bigcap \mathscr{A}$. Since $x \in A$ for every $A \in \mathscr{A}$, we see that $y \in f(A)$ for every $A \in \mathscr{A}$, which implies $y \in \bigcap_{A\in\mathscr{A}} f(A)$.

   For a counterexample to equality, take $A = \{1\}$, $B = \{2\}$ and $f : \{1, 2\} \to \{1\}$ given by $f(1) = f(2) = 1$. Then $A \cap B = \emptyset$ but $f(A) \cap f(B) = \{1\}$.

   If f is injective, then equality holds: take $y \in \bigcap_{A\in\mathscr{A}} f(A)$. Then $y \in f(A)$ for all $A \in \mathscr{A}$, which means that for every $A \in \mathscr{A}$ there is $x_A \in \mathscr{A}$ such that $y = f(x_A)$. Since f is injective, all the x_A must be equal to one another. Call this common value x. We have then $x \in \bigcap \mathscr{A}$ and $y = f(x)$. This shows that $y \in f(\bigcap \mathscr{A})$.

   For an example where equality fails take $X = Y = \{1, 2\}$ and $f(1) = f(2) = 1$. Then $f(\{1\} \cap \{2\}) = f(\emptyset) = \emptyset$, while $f(\{1\}) \cap f(\{2\}) = \{1\}$.

3. Using the definitions of union and inverse image we have that:

$$\begin{aligned} x \in f^{-1}(\bigcup \mathscr{B}) &\Leftrightarrow f(x) \in \bigcup \mathscr{B} \\ &\Leftrightarrow f(x) \in B \text{ for some } B \in \mathscr{B} \\ &\Leftrightarrow x \in f^{-1}(B) \text{ for some } B \in \mathscr{B} \\ &\Leftrightarrow x \in \bigcup_{B\in\mathscr{B}} f^{-1}(B). \end{aligned}$$

4. Using the definitions of intersection and inverse image we have that:

$$\begin{aligned} x \in f^{-1}(\bigcap \mathscr{B}) &\Leftrightarrow f(x) \in \bigcap \mathscr{B} \\ &\Leftrightarrow f(x) \in B \text{ for all } B \in \mathscr{B} \\ &\Leftrightarrow x \in f^{-1}(B) \text{ for all } B \in \mathscr{B} \\ &\Leftrightarrow x \in \bigcap_{B\in\mathscr{B}} f^{-1}(B). \end{aligned}$$

5. Take $x \in A$. Then $f(x) \in f(A)$ (by definition of the direct image $f(A)$), so $x \in f^{-1}(f(A))$ (by definition of the inverse image). This shows that $A \subseteq f^{-1}(f(A))$.

   Let's see that the reverse inclusion (and hence equality) holds if f is injective: assuming this, take $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$, which means that there is $x' \in A$ such that $f(x) = f(x')$. By injectivity, $x = x'$, so $x \in A$.

   For an example where equality does not hold take the same f as in point 2 above and notice that $f^{-1}(f(\{1\})) = f^{-1}(\{1\}) = \{1, 2\} \neq \{1\}$.

6. Take $y \in f(f^{-1}(B))$. Then there is some $x \in f^{-1}(B)$ such that $f(x) = y$. But $x \in f^{-1}(B)$ means that $f(x) \in B$, so $y \in B$.

   The reverse inclusion holds if f is surjective: take $y \in B$. Since f is surjective, there is $x \in X$ such that $f(x) = y \in B$. Then $x \in f^{-1}(B)$, and hence $y \in f(f^{-1}(B))$.

   For an example where equality does not hold take again the same f as in point 2 above. Then $\{2\} \neq f(f^{-1}(\{2\})) = f(\emptyset) = \emptyset$.

Exercise 1.4. Let X be a topological space and let A, U and C be subsets of X. Prove the following:

1. A° is the largest open subset contained in A, $\overline{A}$ is the smallest closed set which contains A and $A^\circ \subseteq A \subseteq \overline{A}$.

2. U is open if and only if $U^\circ = U$ and C is closed if and only if $\overline{C} = C$.

3. If $A \subseteq C$ and C is closed, then $\overline{A} \subseteq C$.

4. $A^{\circ\circ} = A^\circ$ and $\overline{\overline{A}} = \overline{A}$.

5. $\overline{A} = A \cup \{x \in X : x \text{ is a limit point of } A\}$.

Solution. Recall that we are using the following definitions of interior and closure, the ones given in class:

$$\operatorname{int}(A) := \bigcup_{\substack{U \text{ open} \\ U \subseteq A}} U, \qquad \operatorname{cl}(A) := \bigcap_{\substack{C \text{ closed} \\ C \supseteq A}} C.$$

1. $\operatorname{int}(A)$ is open because it is the union of open sets, and it is contained in A because it is the union of subsets of A. Take an open set $U \subseteq A$. Then since U is one of the sets whose union is taken in the definition of $\operatorname{int}(A)$, we have that $U \subseteq \operatorname{int} A$. Hence $\operatorname{int}(A)$ is the largest open set contained in A.

   $\operatorname{cl}(A)$ is closed because it is the intersection of closed sets, and it contains A because it is the intersection of sets which contain A. (Alternatively, one can show that $X \setminus \operatorname{cl}(A)$ is open by using De Morgan's laws:

   $$X \setminus \operatorname{cl}(A) = X \setminus \bigcap_{\substack{C \text{ closed} \\ C \supseteq A}} C = \bigcup_{\substack{C \text{ closed} \\ C \supseteq A}} (X \setminus C),$$

   a union of open sets). Take a closed set $C \supseteq A$. Then since C is one of the sets whose intersection is taken in the definition of $\operatorname{cl}(A)$, we have that $\operatorname{cl}(A) \subseteq C$. Hence $\operatorname{cl}(A)$ is the smallest closed set which contains A.

   The two previous remarks already show that $\operatorname{int}(A) \subseteq A \subseteq \operatorname{cl}(A)$.

2. If $\operatorname{int}(A) = A$ then clearly A is open due to point 1. Conversely, if A is open then A is one of the sets whose union is taken in the definition of $\operatorname{int}(A)$, so $\operatorname{int}(A) \supseteq A$. From point 1 we know that $\operatorname{int}(A) \subseteq A$, so $\operatorname{int}(A) = A$.

   If $\operatorname{cl}(A) = A$ then clearly A is closed due to point 1. Conversely, if A is closed then A is one of the sets whose intersection is taken in the definition of $\operatorname{cl}(A)$, so $\operatorname{cl}(A) \subseteq A$. From point 1 we know that $\operatorname{cl}(A) \supseteq A$, so $\operatorname{cl}(A) = A$.

3. This is a more precise way of saying that "$\operatorname{cl}(A)$ is the smallest closed set that contains A" and was proved in point 1.

4. Since $\text{int}(A)$ is open, $\text{int}(\text{int}(A)) = \text{int}(A)$ by point 2. Since $\text{cl}(A)$ is closed, $\text{cl}(\text{cl}(A)) = \text{cl}(A)$ by point 2.

5. Let us first show that $\text{cl}(A) \supseteq A \cup \{x \in X : x \text{ is a limit point of } A\}$. We know that $A \subseteq \text{cl}(A)$, so we just need to show that if x is a limit point of A then $x \in \text{cl}(A)$. Take an arbitrary closed set $C \supseteq A$. Then $X \setminus C$ is open and $A \cap (X \setminus C) = \emptyset$, so we deduce that $x \notin X \setminus C$ (otherwise we would contradict the definition of a limit point for x). In other words, $x \in C$. We have proved that x must be in every closed set C which contains A, so $x \in \text{cl}(A)$.

   In order to show that $\text{cl}(A) \subseteq A \cup \{x \in X : x \text{ is a limit point of } A\}$, take $x \in \text{cl}(A)$. If $x \in A$ this is obvious, so assume $x \notin A$, and let us show that x is a limit point of A. If U is any open set with $x \in U$, then $X \setminus U$ is a closed set which does not contain x. But since $x \in \text{cl}(A)$, $X \setminus U$ cannot contain A. In other words, $A \cap U \neq \emptyset$, and we have proved that x is a limit point of A.

Exercise 1.6. If A, B are subsets of a topological space X, prove the following identities:

1. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
2. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
3. $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$
4. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$

Give examples in which the inclusions in (b) and (c) above are strict.

Solution. 1. We have $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$. Hence,

$$A \cup B \subseteq \overline{A} \cup \overline{B}.$$

Since $\overline{A} \cup \overline{B}$ is a closed set (being a finite union of closed sets) we have

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}.$$

On the other hand, $A \subseteq A \cup B \subseteq \overline{A \cup B}$. Since $\overline{A \cup B}$ is closed, this implies that

$$\overline{A} \subseteq \overline{A \cup B}.$$

Similarly, $\overline{B} \subseteq \overline{A \cup B}$; taken together, these imply that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

2. Since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, obviously $A \cap B \subseteq \overline{A} \cap \overline{B}$. Since $\overline{A} \cap \overline{B}$ is closed (an intersection of two closed sets) we have that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

3. Since $\text{int}(A) \subseteq A$ and $\text{int}(B) \subseteq B$, obviously $\text{int}(A) \cup \text{int}(B) \subseteq A \cup B$. Since $\text{int}(A) \cup \text{int}(B)$ is open (a union of open sets) we have that $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$.

4. We have $\text{int}(A) \subseteq A$ and $\text{int}(B) \subseteq B$, so $\text{int}(A) \cap \text{int}(B) \subseteq A \cap B$. Since $\text{int}(A) \cap \text{int}(B)$ is open (an intersection of two open sets) we have that $\text{int}(A) \cap \text{int}(B) \subseteq \text{int}(A \cap B)$.

   On the other hand, since $\text{int}(A \cap B) \subseteq A \cap B \subseteq A$ we have that $\text{int}(A \cap B) \subseteq \text{int}(A)$. Analogously, $\text{int}(A \cap B) \subseteq \text{int}(B)$. Hence $\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B)$.

Exercise 1.11. Which of the following collections are bases for a topology on $\mathbb{R}$?

1. $\mathscr{B}_1 = \{(a, b) \mid a, b \in \mathbb{R},\ a < b\}$
2. $\mathscr{B}_2 = \{[a, b] \mid a, b \in \mathbb{R},\ a < b\}$
3. $\mathscr{B}_3 = \{(a, b] \mid a, b \in \mathbb{R},\ a < b\}$
4. $\mathscr{B}_4 = \{[a, b) \mid a, b \in \mathbb{R},\ a < b\}$

Solution. We will use Lemma 4.4 in order to answer the question.

1. Obviously, any point x is in $(x - 1, x + 1)$, so it is contained in at least one element of $\mathscr{B}_1$. On the other hand, if we have $(a, b) \in \mathscr{B}_1$ and $(c, d) \in \mathscr{B}_1$ with nonempty intersection, then their intersection is given by $(\max\{a, c\}, \min\{b, d\})$, which is also an element of $\mathscr{B}_1$, so point 2 of Lemma 4.4 is also satisfied. Hence $\mathscr{B}_1$ is the base of a topology on $\mathbb{R}$ (of course, we know from Example 4.2 that $\mathscr{B}_1$ is indeed a base for the usual topology of $\mathbb{R}$).

2. $\mathscr{B}_2$ does satisfy point 1 of Lemma 4.4, but does not satisfy point 2: if we consider the elements $[1, 2] \in \mathscr{B}_2$ and $[2, 3] \in \mathscr{B}_2$, then $[1, 2] \cap [2, 3] = \{2\}$, and there is no element of $\mathscr{B}_2$ contained in $\{2\}$. Hence $\mathscr{B}_2$ is not a base of any topology on $\mathbb{R}$.

3. Obviously, any point x is in $(x-1, x+1]$, so it is contained in at least one element of $\mathscr{B}_3$. On the other hand, if we have $(a, b] \in \mathscr{B}_3$ and $(c, d] \in \mathscr{B}_3$ with nonempty intersection, then their intersection is given by $(\max\{a, c\}, \min\{b, d\}]$, which is also an element of $\mathscr{B}_3$, so point 2 of Lemma 4.4 is also satisfied. Hence $\mathscr{B}_3$ is the base of a topology on $\mathbb{R}$. (The topology it generates is called *the Sorgenfrey line*; see next exercise.)

4. This is completely analogous to point 3.

Exercise 1.12. The Sorgenfrey line is the set $\mathbb{R}$ with the topology generated by the base $\big\{(a, b] : a < b \in \mathbb{R}\big\}$.

1. Give an example of a set A that is open and closed in the Sorgenfrey line topology but neither open nor closed in the usual Euclidean topology on $\mathbb{R}$.

2. What are the limit points, the interiors and the closures of the following subsets of $\mathbb{R}$ with the Sorgenfrey line topology?
   a) $(0, 1)$; b) $[-\sqrt{2}, \sqrt{2}]$; c) $\{q \in \mathbb{Q} : 0 < q < 1\}$; d) $\{r \in \mathbb{R} - \mathbb{Q} : 0 < r < 1\}$; e) $\{1/n : 1 \le n \in \mathbb{N}\}$; f) $\{-1/n : 1 \le n \in \mathbb{N}\}$; g) $\{\sqrt{2} - 1/n : 1 \le n \in \mathbb{N}\}$.

Solution. 1. $(1,2]$ is one such example: it is open in the Sorgenfrey topology because it belongs to the given base, and closed in the Sorgenfrey line because $\mathbb{R} \setminus (1,2] = (-\infty,1] \cup (2,+\infty)$ is a union of two open sets; but it is neither open nor closed in the usual topology of $\mathbb{R}$.

2. Note that the concepts of limit point, interior and closure can all be stated in terms of a base $\mathscr{B}$ for the topology:

Limit point: x is a limit point of A if and only if for every $B \in \mathscr{B}$ with $x \in B$ it is true that $B \cap A \neq \emptyset$.

Interior: x is in the interior of A if there exists $B \in \mathscr{B}$ with $x \in B \subseteq A$.

Closure: x is in the closure of A if for every $B \in \mathscr{B}$ with $x \in B$ it is true that $A \cap B \neq \emptyset$.

(a) $(0,1)$. Limit points: $(0,1]$. Interior: $(0,1)$. Closure: $(0,1]$.

(b) $[-\sqrt{2},\sqrt{2}]$. Limit points: $[-\sqrt{2},\sqrt{2}]$. Interior: $(-\sqrt{2},\sqrt{2}]$. Closure: $[-\sqrt{2},\sqrt{2}]$.

(c) $(0,1) \cap \mathbb{Q}$. Limit points: $(0,1]$. Interior: $\emptyset$. Closure: $(0,1]$.

(d) $(0,1) \cap (\mathbb{R} \setminus \mathbb{Q})$. Limit points: $(0,1]$. Interior: $\emptyset$. Closure: $(0,1]$.

(e) $\{1/n : 1 \leq n \in \mathbb{N}\}$. Limit points: $\emptyset$. Interior: $\emptyset$. Closure: $\{1/n : 1 \leq n \in \mathbb{N}\}$.

(f) $\{-1/n : 1 \leq n \in \mathbb{N}\}$. Limit points: $\{0\}$. Interior: $\emptyset$. Closure: $\{-1/n : 1 \leq n \in \mathbb{N}\} \cup \{0\}$.

(g) $\{\sqrt{2} - 1/n : 1 \leq n \in \mathbb{N}\}$. Limit points: $\{\sqrt{2}\}$. Interior: $\emptyset$. Closure: $\{\sqrt{2} - 1/n : 1 \leq n \in \mathbb{N}\} \cup \{\sqrt{2}\}$.