MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Problem sheet 1

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Exercise 1.1. 1. Give an example of an infinite subset $\mathscr{A}$ of $\mathscr{P}(\mathbb{N})$ such that $\bigcup_{A\in\mathscr{A}} A = \mathbb{N}$ and $\bigcap_{A\in\mathscr{A}} A = \{17\}$.

2. Give an example of an infinite subset $\mathscr{B}$ of $\mathscr{P}(\mathbb{N})$ such that $\bigcup_{A\in\mathscr{A}} A = \mathbb{N}$ and $\bigcap_{A\in\mathscr{A}} A$ is the set of all prime numbers.

Exercise 1.2. (1 mark for (e) and (f)) Let X and Y be sets and $f : X \to Y$ be a function. Let $A \subseteq X$, $\mathscr{A} \subseteq \mathscr{P}(X)$, $B \subseteq Y$, and $\mathscr{B} \subseteq \mathscr{P}(Y)$. We use the shorthand

$$\bigcup \mathscr{A} \equiv \bigcup_{A\in\mathscr{A}} A,$$

and similarly for $\mathscr{B}$.

1. Prove that $f\big(\bigcup \mathscr{A}\big) = \bigcup_{A\in\mathscr{A}} f(A)$.

2. Prove that $f\big(\bigcap \mathscr{A}\big) \subseteq \bigcap_{A\in\mathscr{A}} f(A)$. Under what conditions on f do you get equality for all collections $\mathscr{A}$? Give an example where equality does not hold?

3. Prove that $f^{-1}\big(\bigcup \mathscr{B}\big) = \bigcup_{B\in\mathscr{B}} f^{-1}(B)$.

4. Prove that $f^{-1}\big(\bigcap \mathscr{B}\big) = \bigcap_{B\in\mathscr{B}} f^{-1}(B)$.

5. Prove that $A \subseteq f^{-1}\big(f(A)\big)$. Under what conditions do you get equality for all sets $A \subseteq X$? Give an example where equality does not hold.

6. Prove that $f\big(f^{-1}(B)\big) \subseteq B$. Under what conditions do you get equality for all sets $B \subseteq Y$? Give an example where equality does not hold.

7. Prove that $X - f^{-1}(B) = f^{-1}(Y - B)$. What can you say about $Y - f(A)$ and $f(X - A)$? Under what conditions would these two sets be equal?

Exercise 1.3. Let $(X, \mathscr{T})$ be a topological space.

1. Prove that a subset $U \subseteq X$ is open if and only if it is a neighbourhood of all of its points.

2. Prove that a subset $C \subseteq X$ is closed if and only if for every $x \in X$, $x \notin C$ there is a neighbourhood of x disjoint from C.

Exercise 1.4. (1 mark for (b), (d) and (e)) Let X be a topological space and let A, U and C be subsets of X. Prove the following:

1. A° is the largest open subset contained in A, $\overline{A}$ is the smallest closed set which contains A and $A^\circ \subseteq A \subseteq \overline{A}$.

2. U is open if and only if $U^\circ = U$ and C is closed if and only if $\overline{C} = C$.

3. If $A \subseteq C$ and C is closed, then $\overline{A} \subseteq C$.

4. $A^{\circ\circ} = A^\circ$ and $\overline{\overline{A}} = \overline{A}$.

5. $\overline{A} = A \cup \{x \in X : x \text{ is a limit point of } A\}$.

Exercise 1.5. Consider the set $\mathbb{Q} \subseteq \mathbb{R}$. Is it open? Is it closed? Describe its interior, its closure, its boundary, and the set of its limit points.

Exercise 1.6. (1 mark) If A, B are subsets of a topological space X, prove the following identities:

1. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

2. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

3. $\operatorname{int}(A) \cup \operatorname{int}(B) \subseteq \operatorname{int}(A \cup B)$

4. $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$

Give examples in which the inclusions in (b) and (c) above are strict.

Exercise 1.7. Consider the sets $X = [0, 1) \cup [2, 3)$ and $Y = [0, 2)$, both with the usual induced topology from $\mathbb{R}$. We define the function $f : X \to Y$ by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ x - 1 & \text{if } 2 \leq x < 3. \end{cases}$$

1. Is the function f continuous?

2. Is it a homeomorphism?

Exercise 1.8. Consider a (nonempty) set X with the trivial topology $\mathscr{T} = \{X, \emptyset\}$. Describe explicitly all possible continuous function from X to R.

Take X a nonempty set. The *discrete metric* on X is the distance d given by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Show that it is a distance and describe explicitly the topology associated to it.

Exercise 1.9. Take two topologies $\mathscr{T}_1$, $\mathscr{T}_2$ on a set X. We say that $\mathscr{T}_1$ is *finer* than $\mathscr{T}_2$ when $\mathscr{T}_2 \subseteq \mathscr{T}_1$ (this is, when $\mathscr{T}_2$ contains less open sets than $\mathscr{T}_1$.) Assume d_1, d_2 are two distances on X such that, for some $C > 0$,

$$d_2(x,y) \leq C d_1(x,y) \quad \text{for all } x, y \in X.$$

Prove that the topology $\mathscr{T}_1$ associated to d_1 is finer than the topology $\mathscr{T}_2$ associated to d_2.

Exercise 1.10. (1 mark) Which of the following collections are bases for a topology on $\mathbb{R}$?

1. $\mathscr{B}_1 = \{(a,b) \mid a, b \in \mathbb{R},\ a < b\}$
2. $\mathscr{B}_2 = \{[a,b] \mid a, b \in \mathbb{R},\ a < b\}$
3. $\mathscr{B}_3 = \{(a,b] \mid a, b \in \mathbb{R},\ a < b\}$
4. $\mathscr{B}_4 = \{[a,b) \mid a, b \in \mathbb{R},\ a < b\}$

Exercise 1.11. (2 marks) The Sorgenfrey line is the set $\mathbb{R}$ with the topology generated by the base $\big\{(a,b] : a < b \in \mathbb{R}\big\}$.

1. Give an example of a set A that is open and closed in the Sorgenfrey line topology but neither open nor closed in the usual Euclidean topology on $\mathbb{R}$.
2. What are the limit points, the interiors and the closures of the following subsets of $\mathbb{R}$ with the Sorgenfrey line topology?
   a) $(0,1)$; b) $[-\sqrt{2}, \sqrt{2}]$; c) $\{q \in \mathbb{Q} : 0 < q < 1\}$; d) $\{r \in \mathbb{R} - \mathbb{Q} : 0 < r < 1\}$; e) $\{1/n : 1 \leq n \in \mathbb{N}\}$; f) $\{-1/n : 1 \leq n \in \mathbb{N}\}$; g) $\{\sqrt{2} - 1/n : 1 \leq n \in \mathbb{N}\}$.

Exercise 1.12. (2 marks) Consider the set $X = \mathbb{R} \cup \{\infty\}$. This is: X is $\mathbb{R}$ with an additional element added, which we call ∞. We define a topology $\mathscr{T}$ on X as follows: a set $U \subseteq X$ is in $\mathscr{T}$ if and only if it is of one of the following types:

1. it is an open set of $\mathbb{R}$,
2. or it is equal to $V \cup \{\infty\}$, with V an open set of $\mathbb{R}$ such that $\mathbb{R} \setminus V$ is bounded.

(When we mention open sets of $\mathbb{R}$ it is understood that we refer to the usual topology.)

1. Prove that $\mathscr{T}$ is a topology on X.
2. Prove that X with the topology $\mathscr{T}$ is homeomorphic to the unit circle of $\mathbb{R}^2$.