MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Solutions to Problem sheet 2

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Unless stated otherwise, in the following exercises X is a topological space with topology $\mathscr{T}$, and A is a subset of X. Also, note that the closure of a set A can be denoted by $\mathrm{cl}(A)$ or $\overline{A}$, and its interior by $\mathrm{int}(A)$ or A°.

Exercise 2.4. Show that the boundary of a set A is empty if and only if A is both open and closed.

Solution. (Note that the definition of "boundary" given in class is $\partial A := \mathrm{cl}(A) \setminus \mathrm{int}(A)$.) If A is both open and closed then by Exercise 1.4 we have $A = \mathrm{int}(A) = \mathrm{cl}(A)$. Hence $\partial A = \mathrm{cl}\, A \setminus \mathrm{int}(A) = A \setminus A = \emptyset$.

Conversely, assume that $\partial A = \mathrm{cl}(A) \setminus \mathrm{int}(A) = \emptyset$. This implies that $\mathrm{cl}(A) \subseteq \mathrm{int}(A)$, and then

$$A \subseteq \mathrm{cl}(A) \subseteq \mathrm{int}(A) \subseteq A.$$

Since we start and end with A in the line above, all the inclusions must be equalities, so $A = \mathrm{cl}(A) = \mathrm{int}(A)$. By Exercise 1.4(2) we deduce that A is both open and closed.

Exercise 2.5. Show that a set U is open if and only if $\partial U = \overline{U} \setminus U$.

Solution. If U is open, then $\mathrm{int}(U) = U$ and obviously $\partial U = \mathrm{cl}(U) \setminus \mathrm{int}(U) = \mathrm{cl}(U) \setminus U$.

Assume now $\partial U = \mathrm{cl}(U) \setminus U$. Since by definition $U = \mathrm{cl}(U) \setminus \mathrm{int}(U)$ and both U and $\mathrm{int}(U)$ are subsets of $\mathrm{cl}(U)$ we deduce that $\mathrm{int}(U) = U$. By Exercise 1.4(2), U is open.

Exercise 2.6. If U is open, is it true that $U = \mathrm{int}(\mathrm{cl}(U))$?

Solution. It is not true in general. A counterexample is given by considering $\mathbb{R}$ with the usual Euclidean topology and $U = (1,2) \cup (2,3)$. Then $\mathrm{int}(\mathrm{cl}(U)) = \mathrm{int}([1,3]) = (1,3)$, not equal to U.

Exercise 2.7. Show that the boundary of a set A (which we defined as $\partial A = \overline{A} \setminus A^\circ$) is equal to $\overline{A} \cap \overline{X \setminus A}$.

Solution. First, notice that $\mathrm{cl}(X \setminus A) = X \setminus \mathrm{int}(A)$; this is because

$$\mathrm{cl}(X \setminus A) = \bigcap_{\substack{C \text{ closed} \\ C \supseteq X\setminus A}} C = \bigcap_{\substack{U \text{ open} \\ U \subseteq A}} (X \setminus U) = X \setminus \bigcap_{\substack{U \text{ open} \\ U \subseteq A}} U = X \setminus \mathrm{int}(A),$$

where we have used De Morgan's laws and the fact that the collection

$$\{C \mid C \text{ closed and } C \supseteq X \setminus A\}$$

is equal to

$$\{X \setminus U \mid U \text{ open and } U \subseteq A\}.$$

Hence we have

$$\mathrm{cl}(A) \cap \mathrm{cl}(X \setminus A) = \mathrm{cl}(A) \cap (X \setminus \mathrm{int}(A)) = \mathrm{cl}(A) \setminus \mathrm{int}(A) = \partial A.$$

Exercise 2.9. Show that a subspace of a Hausdorff space is Hausdorff.

Solution. Consider a Hausdorff topological space X, with topology $\mathscr{T}$, and a subspace $A \subseteq X$ (this is, a subset $A \subseteq X$ with the topology $\mathscr{T}_A$ induced from X). Take two distinct points $x \neq y \in A$. Since X is Hausdorff there exist open sets $U, V \in \mathscr{T}$ such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

Define now $\tilde{U} = U \cap A$, $\tilde{V} = V \cap A$. Then by definition of the topology of A we have $\tilde{U}, \tilde{V} \in \mathscr{T}_A$, and they satisfy

$$x \in \tilde{U}, \quad y \in \tilde{V}, \quad \tilde{U} \cap \tilde{V} = \emptyset.$$

Since x and u were arbitrary, this shows that A is Hausdorff.

Exercise 2.10. Recall that the *cofinite topology* (or *finite complement topology* in a set X is that in which a set is open if and only if it is empty, or its complement has a finite number of points. The cocountable topology is defined analogously (see Handout 1.)

1. Is the cofinite topology of $\mathbb{N}$ Hausdorff? Is the cocountable topology of $\mathbb{N}$ Hausdorff?
2. Is the cofinite topology of $\mathbb{R}$ Hausdorff? Is the cocountable topology of $\mathbb{R}$ Hausdorff?
3. In the cofinite topology of $\mathbb{R}$, which points are limits of the sequence $x_n = 1/n$?

Solution.

1. In the cofinite topology of $\mathbb{N}$ any two nonempty open sets U and V must intersect: by definition there is only a finite number of points not in U, and only a finite number of points not in V; hence, an infinite number of them must be in both. This shows that $\mathbb{N}$ with the cofinite topology is not Hausdorff.

   On the other hand, since $\mathbb{N}$ is countable, the cocountable topology on $\mathbb{N}$ is just the discrete topology (because *every set* has a countable complement), which is always Hausdorff.

2. By the same reasoning as above, the cofinite topology on $\mathbb{R}$ is not Hausdorff. In this case, a similar reasoning applies to the cocountable topology and shows that it is also not Hausdorff: if U, V are any two nonempty open sets they must intersect: by definition there is only a countable number of points not in U, and only a countable number of points not in V; since $\mathbb{R}$ is uncountable, an infinite (in fact uncountable) number of points must be in both.

3. Every point is a limit of $\{1/n\}_{n\geq 1}$. To see this, take any point $x \in \mathbb{R}$ and any open set U with $x \in U$. Since $\mathbb{R} \setminus U$ is at most finite, there must exist $N \in \mathbb{N}$ such that $1/n \notin \mathbb{R} \setminus U$ for all $n \geq N$. Then, $1/n \in U$ for all $n \geq N$, showing that x is a limit of the sequence $\{1/n\}_{n\geq 1}$.

Exercise 2.11. Recall that the *Sorgenfrey line*, or *left limit topology*, is the set $\mathbb{R}$ with the topology generated by the base $\big\{(a, b] : a < b \in \mathbb{R}\big\}$. For each of the following topologies on $\mathbb{R}$, specify which of the others it contains (i.e., which of the others it is finer than):

1. The usual topology.

2. The cofinite topology.

3. The right limit topology.

4. The topology with base given by $\{(a, +\infty) \mid a \in \mathbb{R}\}$.

Solution. Recall that if we are given a base $\mathscr{B}$ for a topology, then a set U is open if and only if for every $x \in U$ there exists $B \in \mathscr{B}$ with $x \in B \subseteq U$. (This is analogous to the way we defined the usual topology of $\mathbb{R}^d$, where the base was the set of open balls.) Using this, it is easy to check that a topology $\mathscr{T}$ is finer than a topology $\mathscr{T}_{\mathscr{B}}$ with base $\mathscr{B}$ if and only if $\mathscr{B} \subseteq \mathscr{T}$.

1. Let us show that the Sorgenfrey topology is finer than the usual topology. Take any set U open in the usual topology. Then for every $x \in U$ there exists $\epsilon > 0$ with $(x - \epsilon, x + \epsilon) \subseteq U$. But then $(x - \epsilon/2, x + \epsilon/2] \subseteq U$. Since x was an arbitrary point of U, we have proved that U is open in the Sorgenfrey topology.

2. The Sorgenfrey topology is finer than the cofinite topology. A way to see this is to notice that any set open in the cofinite topology is also open in the usual topology ($\mathbb{R}$ minus a finite set of points is open in the usual topology), and by the previous point it is also open in the Sorgenfrey topology.

3. The right limit topology and the left limit topology are not comparable (neither is finer than the other): the set $(1, 2]$ is open in the left limit one but not in the right limit one, and the set $[1, 2)$ is open in the right limit one but not in the left limit one.

4. The Sorgenfrey topology is finer than the topology with base $\{(a, +\infty) \mid a \in \mathbb{R}\}$: it is easy to check that every set of the form $(a, +\infty)$ is open in the Sorgenfrey topology.

Exercise 2.12. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *left continuous* if $\lim_{x\to a^-} f(x) = f(a)$ for all $a \in \mathbb{R}$ (this is as the usual concept of continuity, but only for limits *from the left*). Show that a left continuous function is continuous as a function from $\mathbb{R}$ with the left limit topology (see previous problem) to $\mathbb{R}$ with the usual topology.

Solution. Take any set U which is open in the usual topology, and let us prove that $f^{-1}(U)$ is open in the left limit topology. Take $x \in f^{-1}(U)$, and let us show that there exists $\epsilon > 0$ with $(x - \epsilon, x] \subseteq f^{-1}(U)$. (This will be enough to finish the proof due to the observation at the beginning of the solution of Exercise 2.11.)

Since U is open in the usual topology, there exists $\delta > 0$ such that $(f(x)-\delta, f(x)+\delta) \subseteq U$. Using that U is left continuous, there exists $\epsilon > 0$ such that

$$(y < x \text{ and } |x - y| < \epsilon) \Longrightarrow |f(x) - f(y)| < \delta.$$

In other words, $f(y) \in (f(x) - \delta, f(x) + \delta)$. Then for this δ we have $f((x - \epsilon, x]) \subseteq (f(x) - \delta, f(x) + \delta) \subseteq U$, which means that

$$(x - \epsilon, x] \subseteq f^{-1}(U).$$

This shows that $f^{-1}(U)$ is open in the Sorgenfrey topology.