MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Problem sheet 2

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Unless stated otherwise, in the following exercises X is a topological space with topology $\mathscr{T}$, and A is a subset of X.

Exercise 2.1. Let A, B be two subsets of X with $A \subseteq B \subseteq X$. Show that if A is closed in B and B is closed in X, then A is closed in X. (Note: we say that "A is closed in B" to mean that A is closed in the topology of B, which is understood to be the induced from X.)

Exercise 2.2. If $U \subseteq X$ is open and $A \subseteq X$ is closed, show that $U \setminus A$ is open, and $A \setminus U$ is closed.

Exercise 2.3. For $A, B \subseteq X$, show that $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.

Exercise 2.4. (1 mark) Show that the boundary of a set A is empty if and only if A is both open and closed.

Exercise 2.5. (1 mark) Show that a set U is open if and only if $\partial U = \overline{U} \setminus U$.

Exercise 2.6. (1 mark) If U is open, is it true that $U = \operatorname{int}(\operatorname{cl}(U))$?

Exercise 2.7. (1 mark) Show that the boundary of a set A (which we defined as $\partial A = \overline{A} \setminus A^\circ$) is equal to $\overline{A} \cap \overline{X \setminus A}$.

Exercise 2.8. Consider a topological space $(X, \mathscr{T}_X)$ and a subset $A \subseteq X$. Show that the inclusion map $i : A \to X$, $i(x) = x$ is continuous (where the topology in A is understood to be the topology induced from X.)

Exercise 2.9. (1 mark) Show that a subspace of a Hausdorff space is Hausdorff.

Exercise 2.10. (1 mark) Recall that the *cofinite topology* (or *finite complement topology* in a set X is that in which a set is open if and only if it is empty, or its complement has a finite number of points. The cocountable topology is defined analogously (see Handout 1.)

1. Is the cofinite topology of $\mathbb{N}$ Hausdorff? Is the cocountable topology of $\mathbb{N}$ Hausdorff?

2. Is the cofinite topology of $\mathbb{R}$ Hausdorff? Is the cocountable topology of $\mathbb{R}$ Hausdorff?

3. In the cofinite topology of $\mathbb{R}$, which points are limits of the sequence $x_n = 1/n$?

Exercise 2.11. (1 mark) Recall that the *Sorgenfrey line*, or *left limit topology*, is the set $\mathbb{R}$ with the topology generated by the base $\{(a, b] : a < b \in \mathbb{R}\}$. For each of the following topologies on $\mathbb{R}$, specify which of the others it contains (i.e., which of the others it is finer than):

1. The usual topology.

2. The cofinite topology.

3. The right limit topology.

4. The topology with base given by $\{(a, +\infty) \mid a \in \mathbb{R}\}$.

Exercise 2.12. (1 mark) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *left continuous* if $\lim_{x \to a^-} f(x) = f(a)$ for all $a \in \mathbb{R}$ (this is as the usual concept of continuity, but only for limits *from the left*). Show that a left continuous function is continuous as a function from $\mathbb{R}$ with the left limit topology (see previous problem) to $\mathbb{R}$ with the usual topology.