MSM3P22/MSM4P22
Further Complex Variable Theory & General Topology
Problem Sheet 4

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Unless otherwise specified, the symbols X, Y and Z represent topological spaces in the following exercises.

Exercise 4.1. (2 marks) This exercise suggests a way to show that a quotient space is homeomorphic to some other space. Consider an equivalence relation $\sim$ on X, and the quotient topological space $X^* \equiv X/\sim$. Let $f : X \to Y$ be a continuous and surjective map such that

1. For any $x, y \in X$ we have $f(x) = f(y)$ if and only if $x \sim y$.
2. f is open (this is, $f(U)$ is open whenever U is open) *or* f is closed (this is, $f(C)$ is closed whenever C is closed.)

Prove that the map $f^* : X^* \to Y$ given by $f^*(x^*) = f(x)$ for all $x \in X$ (where x^* represents the equivalence class of $x \in X$) is well defined, and is a homeomorphism.

Exercise 4.2. (1 mark) Let X, Y be topological spaces with X compact and Y Hausdorff.

1. Prove that a continuous map $f : X \to Y$ must be closed.
2. Prove that a continuous bijection $f : X \to Y$ must be a homeomorphism.

Exercise 4.3. (1 mark) Show rigorously that $[0, 1]/\{0, 1\}$ is homeomorphic to $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\} \subseteq \mathbb{R}^2$.

Exercise 4.4. (3 marks) Show rigorously that the spaces Y referred to in Exercises 8.11 and 8.12 (Handout 8) are homeomorphic to $X/\mathscr{R}$.

Exercise 4.5. (1 mark) Show that in the finite complement topology of $\mathbb{R}$ (which we also called the cofinite topology), every subset of $\mathbb{R}$ is compact.

Exercise 4.6. (1 mark) In the countable complement topology of $\mathbb{R}$ (which we also called the cocountable topology), is the subset $[0, 1]$ compact?

Exercise 4.7. (1 mark) Show that a finite union of compact subsets of X is compact.

Exercise 4.8. (1 mark)

1. Show that a compact subset of a metric space must be bounded.

2. Find a metric space in which not every closed and bounded subset is compact.

Exercise 4.9. (1 mark) If Y is compact, show that the projection $\pi_X : X \times Y \to X$ is closed (this is, $\pi_X(C)$ is closed in X whenever C is closed in $X \times Y$.)

Exercise 4.10. (1 mark) Consider $\mathbb{R}$ with the left limit topology.

1. Is the interval $[0, 1]$ compact?

2. Is the interval $[0, 1]$ connected?

Exercise 4.11. (1 mark) A collection $\mathscr{C}$ of subsets of X is said to have the *finite intersection property* if every finite subfamily of $\mathscr{C}$ has nonempty intersection. Prove that the following are equivalent:

1. X is compact.

2. Every collection of closed sets in X having the finite intersection property satisfies that the intersection $\bigcap_{C \in \mathscr{C}} C$ of the whole family $\mathscr{C}$ is nonempty.

Exercise 4.12. (2 marks) A *contraction* in a metric space X is a map $f : X \to X$ such that, for some $\alpha < 1$,

$$d(f(x), f(y)) \le \alpha\, d(x, y) \quad \text{for all } x, y \in X.$$

Prove that a contraction on a compact metric space X has a unique fixed point in X. (A *fixed point* of f is a point $x \in X$ such that $f(x) = x$.)

Exercise 4.13. (2 marks) Let X be a sequentially compact metric space. This exercise gives a way to prove that X is compact. Let $\mathscr{A}$ be an open covering of X.

1. Show that there is a number $\delta > 0$ such that for any $x \in X$ there is $U \in \mathscr{A}$ such that $B(x, \delta) \subseteq U$. (This δ is called a *Lebesgue number* for $\mathscr{A}$.)

2. Show that X can be covered by a finite number of balls of radius δ.

3. Deduce that X can be covered by a finite subfamily of $\mathscr{A}$ (which shows that X is compact.)