Evolution equations in a Banach space

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1 Solutions of an evolution equation in an abstract separable Banach space

This notes contain well-known results on the equivalence of several concepts of solution to an evolution equation. They are based on personal notes by Stphane Mischler [2].

We are interested in defining the concept of solution to the following evolution equation in a certain complete separable normed space X:

$$\frac{d}{dt} f = F, \tag{1}$$

where $F : (0,T) \to X$ (for some $0 < T \le +\infty$). We also want to define a solution of the initial value problem

$$\frac{d}{dt} f = F \tag{2}$$

$$f(0) = f^0 \tag{3}$$

for some $f^0 \in X$.[1]

There are many concepts of solution that occur naturally, and in many situations one cannot just stick to one of them and study solutions in that sense. It is good to be able to find solutions with strong differentiability, but it might be difficult to prove their existence, so one usually needs to find solutions in a weaker sense first. Actually, it may happen that some

[1]The theory developed here is valid for a general interval $I \subseteq \mathbb{R}$ instead of $(0,T)$ and a point $x_0 \in \bar{I}$ instead of 0, with minor modifications.

problems have solutions in a weak sense but not strong solutions, so some of the properties and behavior of the equation is lost if we only look at strong solutions.

Here we will always suppose that $F : (0, T) \to X$ is integrable. Under this regularity requirement we will be able to prove that all the concepts or solution of equation (1) or the initial value problem (2) defined below are indeed the same one. We will make use of the theory of integration of functions with values on a Banach space; for an introduction see [1].

Let us first state the different definitions of solution we will consider. In the following, X is a complete separable normed space, $T \in (0, +\infty]$, f is a function $f : (0, T) \to X$ (with no particular regularity assumed) and $F : (0, T) \to X$ is in $L^1((0, T), X)$. Here we mean a concrete function F and not a class of functions in $L^1((0, T), X)$, though the following definitions are the same if F is changed in a set of measure zero. We will denote the norm of X by $\|\cdot\|$.

Some of the definitions below have a distinctive name (such as "mild solution" or "weak solution") and others are stated simply as "solutions". We will always make clear which definition we are talking about when referring to these.

1.1 Definitions of solution to the equation

Definition 1.1 (Mild solution). We say that $f : (0, T) \to X$ is a *mild solution* (or *solution in the sense of semigroups*) to equation (1) if f is continuous in the norm topology and

$$f(t_2) = f(t_1) + \int_{t_1}^{t_2} F(s)\, ds \quad \text{for all } t_1, t_2 \in (0, T). \tag{4}$$

Definition 1.2 (Mild solution, no regularity). We say that $f : (0, T) \to X$ is a solution to equation (1) if

$$f(t_2) = f(t_1) + \int_{t_1}^{t_2} F(s)\, ds \quad \text{for almost all } (t_1, t_2) \in (0, T)^2. \tag{5}$$

Definition 1.3 (Solution in the sense of moments). We say that $f : (0, T) \to X$ is a *solution in the sense of moments* to equation (1) if f is weakly continuous and

$$\langle f(t_2), \phi \rangle = \langle f(t_1), \phi \rangle + \int_{t_1}^{t_2} \langle F(s), \phi \rangle \, ds \quad \text{for all } t_1, t_2 \in (0,T), \quad \phi \in X'. \tag{6}$$

Remark 1.4. By *weakly continuous* we mean that $f : (0,T) \to X$ is continuous when the weak topology in X is considered (and the usual one in $(0,T)$). This is equivalent to the statement that $t \mapsto \langle f(t), \phi \rangle$ is continuous for all $\phi \in X'$.

Remark 1.5. Note that, for all $\phi \in X'$, $s \mapsto \langle F(s), \phi \rangle$ is integrable in $(0,T)$, since $F : (0,T) \to X$ is integrable (see [1], part I, III.2.19 (c)).

Definition 1.6 (Solution in the sense of moments, no regularity). We say that $f : (0,T) \to X$ is a solution to equation (1) if for all $\phi \in X'$ it holds that

$$\langle f(t_2), \phi \rangle = \langle f(t_1), \phi \rangle + \int_{t_1}^{t_2} \langle F(s), \phi \rangle \, ds \quad \text{for almost all } (t_1, t_2) \in (0,T)^2. \tag{7}$$

Definition 1.7. Let $D \subseteq X'$ be dense in the weak-$*$ topology. We say that $f : (0,T) \to X$ is a solution to equation (1) if the conditions in definition 1.6 hold for for all $\phi \in D$ (instead of all $\phi \in X'$).

Definition 1.8 (Weak solution). We say that $f : (0,T) \to X$ is a *weak solution* to equation (1) if for all $\phi \in X'$ we have that $t \mapsto \langle f(t), \phi \rangle$ is locally integrable in $(0,T)$ and

$$\int_0^T \langle f(s), \phi \rangle \frac{d}{ds}\psi(s) \, ds = -\int_0^T \langle F(s), \phi \rangle \, \psi(s) \, ds \quad \text{for all } \phi \in X' \quad \psi \in \mathcal{C}_c^\infty(0,T). \tag{8}$$

Definition 1.9. Let $D \subseteq X'$ be dense in the weak-$*$ topology. We say that $f : (0,T) \to X$ is a solution to equation (1) if the conditions in definition 1.8 hold for for all $\phi \in D$ (instead of all $\phi \in X'$).

1.2 Definitions of solution to the initial value problem

The previous definitions can be easily modified to give definitions of solution to the initial value problem (2); we state them here in the same order as before; note that the conditions in the definitions below clearly include those in the corresponding definition from the previous section.

Definition 1.10 (Mild solution). We say that $f : (0,T) \to X$ is a *mild solution* (or *solution in the sense of semigroups*) to the initial value problem (2) if f is continuous in the norm topology and

$$f(t) = f^0 + \int_0^t F(s)\, ds \quad \text{for all } t \in (0,T). \tag{9}$$

Definition 1.11 (Mild solution, no regularity). We say that $f : (0,T) \to X$ is a solution to the initial value problem (2) if

$$f(t) = f^0 + \int_0^t F(s)\, ds \quad \text{for almost all } t \in (0,T). \tag{10}$$

Definition 1.12 (Solution in the sense of moments). We say that $f : (0,T) \to X$ is a *solution in the sense of moments* to the initial value problem (2) if f is weakly continuous and

$$\langle f(t), \phi \rangle = \left\langle f^0, \phi \right\rangle + \int_0^t \langle F(s), \phi \rangle \; ds \quad \text{for all } t \in (0,T), \quad \phi \in X'. \tag{11}$$

Definition 1.13 (Solution in the sense of moments, no regularity). We say that $f : (0,T) \to X$ is a solution to the initial value problem (2) if for all $\phi \in X'$ it holds that

$$\langle f(t), \phi \rangle = \left\langle f^0, \phi \right\rangle + \int_0^t \langle F(s), \phi \rangle \; ds \quad \text{for almost all } t \in (0,T). \tag{12}$$

Definition 1.14. Let $D \subseteq X'$ be dense in the weak-$*$ topology. We say that $f : (0,T) \to X$ is a solution to the initial value problem (2) if the conditions in definition 1.13 hold for for all $\phi \in D$ (instead of all $\phi \in X'$).

Definition 1.15 (Weak solution). We say that $f : (0,T) \to X$ is a *weak solution* to the initial value problem (1) if for all $\phi \in X'$ we have that $t \mapsto \langle f(s), \phi \rangle$ is locally integrable in $(0,T)$ and

$$\int_0^T \langle f(s), \phi \rangle \frac{d}{ds}\psi(s)\, ds = -\left\langle f^0, \phi \right\rangle \psi(0) - \int_0^T \langle F(s), \phi \rangle\, \psi(s)\, ds$$
$$\text{for all } \phi \in X' \quad \psi \in \mathcal{C}_c^1([0,T)). \tag{13}$$

Definition 1.16. Let $D \subseteq X'$ be dense in the weak-$*$ topology. We say that $f : (0,T) \to X$ is a solution to the initial value problem (2) if the conditions in definition 1.15 hold for for all $\phi \in D$ (instead of all $\phi \in X'$).

The following definitions are easily seen to be equivalent to definitions 1.10 and 1.12, respectively:

Definition 1.17. We say that $f : (0,T) \to X$ is a *mild solution* (or *solution in the sense of semigroups*) to the initial value problem (2) if it is a mild solution to equation (1) and $\|f(t) - f^0\| \to 0$ when $t \to 0^+$.

Definition 1.18. We say that $f : (0,T) \to X$ is a *solution in the sense of moments* to the initial value problem (2) if it is a solution in the sense of moments to equation (1) and $f(t) \rightharpoonup f^0$ in the weak topology when $t \to 0^+$.

1.3 Existence of solutions

Theorem 1.19. *There exists a mild solution f to equation (1), given by the primitive of F:*

$$f(t) := \int_0^t F(s)\, ds.$$

Another function g is a mild solution if and only if it differs from f by a constant: for some $x \in X$ it happens that $f(t) - g(t) = x$ for all $t \in (0,T)$.

The only mild solution to the initial value problem (2) is given by

$$h(t) := f^0 + \int_0^t F(s)\, ds.$$

Proof. It is obvious from the properties of the integral that the f defined in the statement is a mild solution to equation (1). If g is any other mild solution, fix $t_1 \in (0,T)$; it is clear from (4) written for f and g that the difference between $f(t_2)$ and $g(t_2)$ is $f(t_1) - g(t_1)$ for all $t_2 \in (0,T)$; conversely, adding a constant in X to a solution gives another solution. Hence, the function h defined in the theorem is a solution and also a mild solution of the initial value problem (2) (according to definition 1.10).

The solution to the initial value problem is unique, as we have proved that any other solution differs from it by a constant $x \neq 0$ and so does not satisfy equation (9). □

1.4 Equivalence of the definitions of solution

Under the previous assumptions, these definitions are equivalent: a solution in the sense of any of them is also a solution in the sense of all the others, possibly after being changed in a set of measure zero. The key assumption is the regularity of the function F in equation (1); some of these solutions make sense when F is less regular and then it may happen that not all of these concepts are equivalent; however, they are when F is integrable. Let us prove this.

Theorem 1.20 (Equivalence of the concepts of solution to the equation). *If a function $f : (0, T) \to X$ is a solution to equation (1) in the sense of any of the previous definitions, then it can be modified in a set of measure zero so that it becomes a solution to equation (1) in the sense of* all *of the previous definitions.*

In the following proof we say that a given definition implies some other if a solution in the sense of the former must also be a solution in the sense of the latter. Though it looks long, the only difficulty in it lies in going "up hill" to show that a weak solution is a.e. equal to a strong solution; the rest — proving that every solution implies the next one (in the order given here) — is done by means of straightforward arguments.

Proof. Definition 1.1 includes definition 1.2. Conversely, if f is a solution in the sense of definition 1.2, then fix $t_1 \in (0, T)$ such that (5) holds for almost all $t_2 \in (0, T)$; we see that $f(t)$ coincides with $\tilde{f}(t) := f(t_1) + \int_{t_1}^{t} F(s)\, ds$ for almost all $t \in (0, T)$, and this latter function is a mild solution as we know from theorem 1.19.

A solution according to definition 1.1 is also a solution according to 1.3: the regularity of f is already given, and (6) can be obtained by applying ϕ to the equality (4) (recall that $\left\langle \int_{t_1}^{t_2} F(s)\, ds, \phi \right\rangle = \int_{t_1}^{t_2} \langle F(s), \phi \rangle \, ds$; see for example [1], part I, III.2.19 (c)).

Definition 1.3 clearly implies definition 1.6; definition 1.6 implies 1.7. Actually, definition 1.7 also implies definition 1.6; this can be seen by taking a sequence $\{\phi_n\}$ of elements in D that converges weak-$*$ to a given $\phi \in X'$. One can pass to the limit in equation (7) (written for ϕ_n) by using the usual dominated convergence theorem for real functions, as $\|\phi_n\|$ is bounded by some constant $K > 0$ and thus $|\langle F(s), \phi \rangle| \leq K \|F(s)\|$, which is integrable in s.

It is easy to see that definition 1.6 implies definition 1.8: let f be a solution according to definition 1.6. Pick any $\phi \in X'$ and fix $t_1 \in (0, T)$ so that (7) holds for almost all $t_2 \in (0, T)$. This shows that $t_2 \mapsto \langle f(t_2), \phi \rangle$ is locally integrable in $(0,T)$ (actually, it is equal a.e. to the right hand side, a continuous function).

Now take any $\psi \in \mathcal{C}^1_c(0,T)$ and fix $t_1 < \inf \operatorname{supp} \phi$ such that (7) holds for almost all $t_2 \in (0,T)$. Multiply (7) by $\frac{d}{dt}\psi$ and integrate:

$$\begin{aligned}
&\int_0^T \langle f(s), \phi \rangle \frac{d}{dt}\psi(s)\, ds \\
&\qquad = \langle f(t_1), \phi \rangle \int_0^T \frac{d}{dt}\psi(s)\, ds + \int_0^T \int_{t_1}^s \langle F(\tau), \phi \rangle \frac{d}{dt}\psi(s)\, d\tau\, ds \\
&\qquad = \int_{t_1}^T \int_{t_1}^s \langle F(\tau), \phi \rangle \frac{d}{dt}\psi(s)\, d\tau\, ds \\
&\qquad = \int_{t_1}^T \int_{\tau}^T \langle F(\tau), \phi \rangle \frac{d}{dt}\psi(s)\, ds\, d\tau \\
&\qquad = \int_{t_1}^T \langle F(\tau), \phi \rangle \int_{\tau}^T \frac{d}{dt}\psi(s)\, ds\, d\tau \\
&\qquad = -\int_{t_1}^T \langle F(\tau), \phi \rangle\, \psi(\tau)\, d\tau = -\int_0^T \langle F(\tau), \phi \rangle\, \psi(\tau)\, d\tau.
\end{aligned}$$

Definition 1.8 evidently implies 1.9; 1.9 also implies 1.8, as we can use the same argument used to prove that definition 1.7 implies definition 1.6.

Let us gather what we have proved. If we write "definition A implies definition B" as $A \implies B$ and "a solution in the sense of definition A is a.e. equal to a solution in the sense of definition B" as $A \dashrightarrow B$, then we have that $1.1 \implies 1.3 \implies 1.6 \implies 1.8$. Also, $1.1 \implies 1.2 \dashrightarrow 1.1$, $1.6 \Leftrightarrow 1.7$ and $1.8 \Leftrightarrow 1.9$. If we prove that a solution in the sense of definition 1.8 is equal to a solution in the sense of definition 1.1 a.e. ($1.8 \dashrightarrow 1.1$), then the theorem is proved.

So, let f be a weak solution to equation (1), and take $\phi \in X'$. Equation (8) means that

$$\frac{d}{dt} \langle f(t), \phi \rangle = \langle F(t), \phi \rangle \quad \text{in } \mathcal{D}'(0,T).$$

As $t \mapsto \langle F(t), \phi \rangle$ is integrable on $(0,T)$, from the theory of distributions we know that there exists $C \in \mathbb{R}$ and a set $E_\phi \subseteq (0,T)$ such that $(0,T) \setminus E_\phi$ has measure zero and

$$\langle f(t), \phi \rangle = C + \int_0^t \langle F(s), \phi \rangle\, ds \quad \text{for all } t \in E_\phi.$$

Hence, for all $(t_1, t_2) \in E_\phi \times E_\phi$,

$$\langle f(t_2), \phi \rangle = \langle f(t_1), \phi \rangle + \int_{t_1}^{t_2} \langle F(s), \phi \rangle \, ds.$$

(We already have definition 1.6). The problem is that the set E_ϕ depends on ϕ; if it was true for all ϕ and a fixed set E, then the last equality would imply (5) (if $a, b \in X$ are such that $\langle a, \phi \rangle = \langle b, \phi \rangle$ for all $\phi \in X'$ then $a = b$; apply this for $a = f(t_2)$, $b = f(t_1) + \int_{t_1}^{t_2} F(s)\, ds$). To overcome this, take $\{\phi_n\}_{n \in \mathbb{N}}$ a countable subset of X' which is dense in the weak-$*$ topology on X'. This can be done because when X is separable (for the norm topology), X' is separable for the weak-$*$ topology (actually, every ball in X' is weak-$*$ compact, and every ball is also metrizable; hence, the weak-$*$ topology in every ball is separable; see [1, theorems V.4.2, V.5.1, I.6.19]).[2] Then the argument above, applied to each ϕ_n, proves that with

$$E := \bigcap_{n \in \mathbb{N}} E_{\phi_n}$$

we have that

$$\langle f(t_2), \phi_n \rangle = \langle f(t_1), \phi_n \rangle + \int_{t_1}^{t_2} \langle F(s), \phi_n \rangle \, ds \quad \text{for all } n \in \mathbb{N}, \quad (t_1, t_2) \in E \times E. \tag{14}$$

Note that, E being the intersection of countably many sets whose complements have measure zero, $(0, T) \setminus E$ has measure zero. Now, for any $\phi \in X'$ we can choose a sequence $\{\phi_{n(m)}\}_{m \in \mathbb{N}}$ which converges weak-$*$ to ϕ and pass to the limit in (14) with the help of the dominated convergence theorem (as we did earlier in the proof) to obtain that

$$\langle f(t_2), \phi \rangle = \langle f(t_1), \phi \rangle + \int_{t_1}^{t_2} \langle F(s), \phi \rangle \, ds \quad \text{for all } \phi \in X', \quad (t_1, t_2) \in E \times E.$$

This does imply, as explained above, the conditions in definition 1.2, and so f coincides a.e. with a solution in the sense of definition 1.1. This finishes the proof. $\square$

[2] Another way to see this: one can show that, given $\{x_n\}$ a countable dense subset of X, the family of sets

$$\{\phi \in X' \mid s < \langle \phi, x_{n_i} \rangle < r \text{ for } i = 1, \dots, k\} \quad \text{for } r, s \in \mathbb{Q}, \quad k \in \mathbb{N}, \quad n_1, \dots, n_k \in \mathbb{N}$$

forms a countable base for the weak-$*$ topology in X'.

1.5 Equivalence of the definitions of solution to the initial value problem

Theorem 1.21. *If a function $f : (0,T) \to X$ is a solution to the initial value problem (2) in the sense of any of the definitions in section 1.2, then it can be modified in a set of measure zero so that it becomes a solution to the initial value problem in the sense of* all *of the definitions in section 1.2.*

Proof. If f is a solution to the initial value problem (2) in the sense of any of our definitions, then in particular it is a.e. equal to a solution to equation (1) in the sense of *any* of the definitions 1.1–1.9, thanks to theorem 1.20. Hence, it is a.e. equal to a solution which is continuous on $(0,T)$, which we still denote by f. It is enough then to prove that a continuous function $f : (0,T) \to X$ which is a solution to the i.v.p. in the sense of any of the definitions 1.10–1.16 is also a solution to the i.v.p. in the sense of all the others.

It is easy to see, using very similar arguments to those in the proof of theorem 1.20, that if f is continuous in the norm topology then every definition implies the next one, in the order given. The only difficulty is to show that if f satisfies 1.16, then it satisfies 1.10. Let us prove this.

Observe first that f can be extended continuously to $[0,T)$. Take any decreasing sequence $t_n \to 0$. As f is a mild solution to the equation (1), for any $m > n$ we have

$$\|f(t_n) - f(t_m)\| \leq \int_{t_m}^{t_n} \|F(s)\| \, ds \leq \int_0^{t_n} \|F(s)\| \, ds,$$

which tends to 0 when $n \to \infty$. Hence, $f(t_n)$ is a Cauchy sequence, so it is convergent in X (as X is complete). As this is true for any decreasing sequence $t_n \to 0$, we know that $f(t)$ has a limit when $t \to 0$. Define $f(0)$ as this limit; this makes $f : [0,T) \to X$ continuous. Taking the limit when $t_1 \to 0$ in (4) we obtain that

$$f(t) = f(0) + \int_0^t F(s) \, ds \quad \text{for all } t \in (0,T),$$

so f is a solution of the initial value problem (2) with initial value $f(0)$; we need to show that $f(0) = f^0$. But we know definition 1.10 implies the rest of definitions, so in particular we have that

$$\int_0^T \langle f(s), \phi \rangle \frac{d}{ds}\psi(s)\, ds = -\langle f(0), \phi \rangle \psi(0) - \int_0^T \langle F(s), \phi \rangle \psi(s)\, ds$$
$$\text{for all } \phi \in D \quad \psi \in \mathcal{C}^1_c([0,T)).$$

As f is a solution of the initial value problem in the sense of definition 1.16, the same is true with f^0 instead of $f(0)$. Hence,

$$\langle f(0), \phi \rangle = \left\langle f^0, \phi \right\rangle \qquad \text{for all } \phi \in D.$$

This proves that $f(0) = f^0$. □

2 Solutions of an evolution equation in L^1

Let Ω be an open subset of $\mathbb{R}^N$ and μ a positive Borel measure on Ω. We can put $X = L^1(\Omega, \mu)$ (which we denote as $L^1(\Omega)$, understanding the measure μ) in the previous section and obtain several definitions of solution of an evolution equation in $L^1(\Omega)$, which have been proved to be equivalent when the F in equation (1) is regular enough. Here we want to particularize the definitions in this case and add another one when μ is finite on compact sets, that of *renormalized solution*, which does not have a direct analogy in an abstract Banach space.

Of course, definitions in the previous section *do* directly apply to the case $X = L^1(\Omega)$, but it is sometimes more convenient to phrase them in slightly different terms: equality between functions in L^1 is usually expressed as equality a.e., and an integrable function $F : (0,T) \to L^1(\Omega)$ is more commonly regarded as a real integrable function on $(0,T) \times \Omega$. We start by stating this latter relationship precisely, after [1, theorem III.11.16]:

Theorem 2.1. *Let $0 < T \leq +\infty$ and $(S, \mathcal{A}, \mu)$ be a positive measure space. We consider the Lebesgue measure dt on $(0,T)$ and the product measure $dt \otimes \mu$ on $(0,T) \times S$.*

1. *If $\tilde{f} : (0,T) \to L^1(S)$ is integrable, then there exists an integrable function $f : (0,T) \times S \to \mathbb{R}$ such that $f(t, \cdot) = \tilde{f}(t)$ for almost all $t \in (0,T)$.*

2. *Let* $f : (0,T) \times S \to \mathbb{R}$ *be an integrable function. Then the function* $\tilde{f} : (0,T) \to L^1(S)$ *defined for almost all* $t \in (0,T)$ *by* $\tilde{f}(t) := f(t,\cdot)$ *is integrable.*

In any of these cases $\int_0^T f(t,x)\,dt$ *(which exists for almost all* $x \in S$*) is a.e. equal to* $\int_0^T \tilde{f}(t)\,dt$*.*

This enables us to speak interchangeably of integrable functions from $(0,T)$ to $L^1(\Omega)$ and integrable functions on $(0,T) \times \Omega$.

Now we state definitions 1.1–1.9 in this particular case. Definitions of solution of the initial value problem are the analogous modification of the definitions in section 1.2 and are not explicitly given.

Below, $\Omega \subseteq \mathbb{R}^N$ is an open set and μ is a positive Borel measure on Ω which is finite on compact sets.[3] We always consider the Lebesgue measure dt on $(0,T)$, the measure μ on Ω and the product measure $dt \otimes \mu$ on $(0,T) \times \Omega$. All integrals are taken with respect to these measures; integrals with respect to μ will be indicated with $d\mu(x)$. As before, $T \in (0,+\infty]$. Now, $F : (0,T) \times \Omega \to \mathbb{R}$ is a function in $L^1((0,T) \times \Omega)$, and again we mean a concrete function F and not a class of functions in $L^1((0,T) \times \Omega)$.

Definition 2.2 (Mild solution). We say that an integrable function $f : (0,T) \times \Omega \to \mathbb{R}$ is a *mild solution* (or *solution in the sense of semigroups*) to equation (1) if f is continuous in the norm topology (when viewed as a function $f : (0,T) \to L^1(\Omega)$ as specified in theorem 2.1) and for all $t_1, t_2 \in (0,T)$ one has that

$$f(t_2,x) = f(t_1,x) + \int_{t_1}^{t_2} F(s,x)\,ds \quad \text{for almost all } x \in \Omega. \tag{15}$$

Definition 2.3 (Mild solution, no regularity). We say that $f : (0,T) \times \Omega \to \mathbb{R}$ is a solution to equation (1) if $f(t,\cdot)$ is integrable for almost all $t \in (0,T)$ and for almost all $(t_1,t_2) \in (0,T)^2$ one has that

$$f(t_2,x) = f(t_1,x) + \int_{t_1}^{t_2} F(s,x)\,ds \quad \text{for almost all } x \in \Omega. \tag{16}$$

[3]The condition that μ be finite on compact sets is imposed so that we can integrate all continuous functions of compact support on Ω, a fact which is used later to define weak solutions.

Definition 2.4 (Solution in the sense of moments). We say that $f : (0,T) \times \Omega \to \mathbb{R}$ is a *solution in the sense of moments* to equation (1) if $f(t,\cdot)$ is integrable for all $t \in (0,T)$, f is weakly continuous (when viewed as a function $f : (0,T) \to L^1(\Omega)$) and

$$\int_\Omega f(t_2,x)\phi(x)\,d\mu(x) = \int_\Omega f(t_1,x)\phi(x)\,d\mu(x) + \int_{t_1}^{t_2}\int_\Omega F(s,x)\phi(x)\,d\mu(x)\,ds \quad \text{for all } t_1,t_2 \in (0,T), \quad \phi \in L^\infty(\Omega). \tag{17}$$

Definition 2.5 (Solution in the sense of moments, no regularity). We say that $f : (0,T) \times \Omega \to \mathbb{R}$ is a *solution in the sense of moments* to equation (1) if $f(t,\cdot)$ is integrable for almost all $t \in (0,T)$ and

$$\int_\Omega f(t_2,x)\phi(x)\,d\mu(x) = \int_\Omega f(t_1,x)\phi(x)\,d\mu(x) + \int_{t_1}^{t_2}\int_\Omega F(s,x)\phi(x)\,d\mu(x)\,ds \quad \text{for almost all } (t_1,t_2) \in (0,T)^2, \quad \phi \in L^\infty(\Omega). \tag{18}$$

Definition 2.6. Let $D \subseteq L^\infty(\Omega)$ be dense in the weak-$*$ topology. We say that $f : (0,T) \times \Omega \to \mathbb{R}$ is a solution to equation (1) if the conditions in definition 2.5 hold for all $\phi \in D$ (instead of all $\phi \in L^\infty(\Omega)$).

Definition 2.7 (Weak solution). We say that $f : (0,T) \times \Omega \to \mathbb{R}$ is a *weak solution* to equation (1) if $f(t,\cdot)$ is integrable for almost all $t \in (0,T)$, f is locally integrable in $(0,T) \times \Omega$ and

$$\int_0^T \int_\Omega f(s,x)\,\partial_t\varphi(s,x)\,d\mu(x)\,ds = -\int_0^T \int_\Omega F(s,x)\,\varphi(s,x)\,d\mu(x)\,ds \quad \text{for all } \varphi \in \mathcal{D}((0,T)\times\Omega). \tag{19}$$

Remark 2.8. It is not difficult to see that this definition is equivalent to definition 1.9 in our present case ($X = L^1(\Omega)$): first, note that all functions in $\mathcal{D}((0,T)\times\Omega)$ are $dt \otimes \mu$-integrable (as μ is a Borel measure, and finite on compact sets). If we take φ of the form $\varphi(t,x) = \phi(x)\psi(t)$ for ϕ, ψ $\mathcal{C}^\infty$ and of compact support, then one sees that the previous definition implies definition 1.9, as $\mathcal{D}(\Omega)$ is weak-$*$ dense in $L^\infty(\Omega)$ (seen as the dual space of $L^1(\Omega,\mu)$). Conversely, definition 1.9 implies this one: if we call Y the set spanned by functions φ of the form $\varphi(t,x) = \phi(x)\psi(t)$ with $\phi \in \mathcal{D}(\Omega), \psi \in \mathcal{D}(0,T)$, then one can uniformly approximate functions in $\mathcal{D}((0,T)\times\Omega)$ by functions in Y so that $\partial_t\varphi$ is also uniformly approximated.

Remark 2.9. When defining solutions to the initial value problem, the condition that $f(t, \cdot)$ is integrable for almost all $t \in (0, T)$ can be omitted in definitions 2.3 and 2.7, as it is implied by the fact that the initial condition is in $L^1(\Omega)$ and $F : (0, T) \times \Omega \to \mathbb{R}$ is integrable.

2.1 Renormalized solutions

The following definition is new:

Definition 2.10 (Renormalized solution). We say that a measurable function $f : (0, T) \times \Omega \to \mathbb{R}$ is a *renormalized solution* of equation (1) if, in the sense of distributions in $(0, T) \times \Omega$,

$$\frac{d}{dt}\beta(f) = \beta'(f)F \quad \text{for all } \beta \in \mathcal{C}^{1,b}(\mathbb{R}). \tag{20}$$

Remark 2.11. The notation $\mathcal{C}^{1,b}(A)$ represents the set of all bounded functions with continuous and bounded first-order derivatives in a set $A \subseteq \mathbb{R}^N$.

Remark 2.12. Note that the expressions in the definition make sense: β being continuous and bounded, $\beta(f)$ is measurable and bounded, so it is a distribution; for similar reasons $\beta'(f)$ is in $L^\infty((0, T) \times \Omega)$, so $\beta'(f)F$ is integrable on $(0, T) \times \Omega$ and in particular is a distribution.

Theorem 2.13. *Let f be a renormalized solution to equation (1). If f is in $L^1((0, T) \times \Omega)$, then f is almost everywhere equal to a solution to (1) in the sense of all of our previous definitions.*

Proof of theorem 2.13. We will prove that a renormalized solution f which is also integrable must be a weak solution (which implies the result thanks to theorem 1.20. Equation (20) means that for all $\varphi \in \mathcal{D}((0, T) \times \Omega)$

$$-\int_0^T \int_\Omega \beta(f(s, x))\partial_t\varphi(s, x)\, d\mu(x)\, ds = \int_0^T \int_\Omega \beta'(f(s, x))F(s, x)\varphi(s, x)\, d\mu(x)\, ds. \tag{21}$$

Take $\beta_n(s) = \int_0^s \rho(\frac{y}{n})\, dy$, with ρ a $\mathcal{C}^{1,b}$ function which is 1 on $[-1, 1]$, 0 outside $[-2, 2]$ and is always between 0 and 1. Then obviously $|\beta_n(s)| \leq |s|$. Now, $\beta_n'(f)$ converges pointwise to the constant 1 function and is uniformly bounded in n. By the dominated convergence theorem, this enables us to pass to the limit in the right hand side of equation (21) written for β_n instead of β. To pass to the limit in the left hand side, note that $\beta_n(f)$ converges pointwise to f and $|\beta_n(f)| \leq |f|$, which is integrable by hypothesis. Then, the dominated convergence theorem finishes the proof. $\square$

Remark 2.14. The condition that f is integrable *is necessary*. For an example of a renormalized solution which is *not* a weak or mild solution, take an F which is, say, differentiable and of compact support, and consider a mild solution f to (1), so that f is also differentiable. Then define

$$\tilde{f}(t,x) := f(t,x) + h(x) \quad \text{ for all } (t,x) \in (0,T) \times \Omega,$$

where $h : \Omega \to \mathbb{R}$ is any *non-integrable* differentiable function. Then $\tilde{f}$ is not integrable (so it is not a mild solution), but it is still a renormalized solution. It is easy to prove this, as in this case we can operate directly. For any $\phi \in \mathcal{C}_c^\infty((0,T) \times \Omega)$ we have:

$$\begin{aligned}\int_0^T \int_\Omega \beta(\tilde{f}) \partial_t \phi \, d\mu(x) \, dt &= - \int_0^T \int_\Omega \partial_t \beta(\tilde{f}) \phi \, d\mu(x) \, dt \\ &= - \int_0^T \int_\Omega \beta'(\tilde{f}) \partial_t (f+h) \phi \, d\mu(x) \, dt = - \int_0^T \int_\Omega \beta'(\tilde{f}) F \phi \, d\mu(x) \, dt.\end{aligned}$$

Nevertheless, here we do not need to care much about the fact that renormalized solutions may not be solutions according to the other definitions, as the distinction is not important for our purposes. In fact, we could have included the condition that f be integrable in the definition of renormalized solution; it was not done this way so that the definition is simpler and does not include any apparently unnecessary conditions.

Theorem 2.15. *Let f be a solution to equation in the sense of any of the definitions 2.2– 2.7. Then, f is also a renormalized solution to equation (1).*

Proof. As we know all definitions are equivalent, it is enough to prove that a solution to equation (1) in the sense of definition 2.2 is also a renormalized solution. To do this we will regularize our equation and then pass to the limit, but let us first extend f and F to $\mathbb{R} \times \Omega$ so that calculations become easier later: extend F to $\mathbb{R} \times \Omega$ by zero and define

$$\begin{aligned} f(t,\cdot) &:= f(T,\cdot) && \text{for } t \geq T, \\ f(t,\cdot) &:= f(0,\cdot) && \text{for } t \leq 0. \end{aligned}$$

Note that f, thus extended, is a solution to $\frac{d}{dt}f = F$ on $\mathbb{R}$ (not only on $(0,T)$). Now take ρ_n to be a regularizing sequence on $\mathbb{R}$ (for example, $\rho_n(t) := \frac{1}{\epsilon}\rho(\frac{1}{\epsilon}t)$ with ρ a nonnegative $\mathcal{C}^\infty$ function with integral 1) and define for $(t,x) \in \mathbb{R} \times \Omega$:

$$F_n(t,x) := (F *_t \rho_n)(t,x) = \int_{-\infty}^{\infty} F(s,x)\rho_n(t-s)\,ds = \int_{-\infty}^{\infty} F(t-s,x)\rho_n(s)\,ds,$$

$$f_n(t,x) := (f *_t \rho_n)(t,x) = \int_{-\infty}^{\infty} f(s,x)\rho_n(t-s)\,ds = \int_{-\infty}^{\infty} f(t-s,x)\rho_n(s)\,ds.$$

Then f_n is a solution to equation (1) with F_n instead of F:

$$\begin{aligned} f_n(t_1,x) - f_n(t_2,x) &= \int_{-\infty}^{\infty} (f(t_1-s,x) - f(t_2-s,x))\rho_n(s)\,ds \\ &= \int_{-\infty}^{\infty}\int_{t_2-s}^{t_1-s} F(\tau,x)\rho_n(s)\,d\tau\,ds = \int_{-\infty}^{\infty}\int_{t_2}^{t_1} F(\tau-s,x)\rho_n(s)\,d\tau\,ds \\ &= \int_{t_2}^{t_1}\int_{-\infty}^{\infty} F(\tau-s,x)\rho_n(s)\,ds\,d\tau = \int_{t_2}^{t_1} F_n(\tau,x)\,d\tau. \end{aligned}$$

It is easy to see that $f_n(t,x)$ is differentiable in t for almost all $x \in \Omega$ (as $f(\cdot,x)$ is integrable for almost all $x \in \Omega$). In fact, as f_n is a solution to $\frac{d}{dt}f_n = F_n$, we know that

$$\partial_t f_n(t,x) = G_n(t,x) \quad \text{for almost all } (t,x) \in \mathbb{R} \times \Omega.$$

Hence, we can write the following for any $\beta \in \mathcal{C}^{1,b}(\mathbb{R})$ and almost all $x \in \Omega$ (note that we have omitted the variables (t,x) below to make the expressions more readable):

$$\begin{aligned} -\int_0^T \int_\Omega \beta(f_n)\partial_t\phi\,d\mu(x)\,dt &= \int_0^T \int_\Omega \partial_t\beta(f_n)\phi\,d\mu(x)\,dt \\ &= \int_0^T \int_\Omega \beta'(f_n)\partial_t f_n \phi\,d\mu(x)\,dt = \int_0^T \int_\Omega \beta'(f_n)G_n\phi\,d\mu(x)\,dt. \end{aligned}$$

In order to pass to the limit in the previous expression, first observe that $f_n \to f$ in $L^1((0,T)\times\Omega)$ (which is a common result on approximation by convolution) and so $\beta(f_n) \to \beta(f)$ in the same space, as

$$|\beta(f_n) - \beta(f)| \leq \|\beta'\|_\infty |f_n - f|.$$

This allows us to pass to the limit in the first term. For the last one, as $G_n \to G$ in $L^1((0,T) \times \Omega)$, we only need that

$$\beta'(f_n) \to \beta'(f) \quad \text{weak-}* \text{ in } L^\infty((0,T) \times \Omega).$$

This is true because

- $f_n \to f$ almost uniformly in compact sets (as it converges in L^1),
- β', being continuous, is uniformly continuous in compact sets,
- so $\beta'(f_n) \to \beta'(f)$ almost uniformly in compact sets, which implies weak-$*$ convergence in L^∞.

□

Remark 2.16. One can define a concept of renormalized solution to the initial value problem (2) by imposing that (20) must be satisfied in the sense of distributions on $[0,T) \times \Omega$, with the appropriate boundary term which involves the initial condition. Though we will not prove it here, a renormalized solution to the initial value problem in this sense is always a solution in the sense of the rest of the definitions, as the condition that f be integrable is implied if the initial condition is integrable. In this work, when we talk about solutions of the initial value problem we mean solutions in the sense of any of definitions 1.10– 1.16, which include o imply the condition that f be integrable.

2.2 Some properties of renormalized solutions

In this section, $f \in L^1((0,T) \times \Omega)$ will always be a mild solution to (1) (so it is a solution in the sense of all other definitions, including definition 2.10 of renormalized solutions).

We want to prove that, thanks to the assumed regularity of F, these solutions satisfy (20) in a stronger sense and for more general β than $\mathcal{C}^{1,b}$ functions. To be precise, what we want to prove is the following:

Theorem 2.17. *Let f be a mild solution to the initial value problem (2) in the space $X = L^1(\Omega)$. Then for all piecewise differentiable $\beta : \mathbb{R} \to \mathbb{R}$*

such that $\beta'(f)F \in L^1((0,T)\times\Omega)$ *and* $\beta(f^0) \in L^1(\Omega)$ *it happens that* $\beta(f) : (0,T) \to L^1(\Omega)$ *is continuous in the norm topology and*

$$\beta(f(t)) = \beta(f^0) + \int_0^t \beta'(f(s))F(s)\,ds \quad \text{for all } t \in (0,T). \tag{22}$$

(This is, $\beta(f)$ *is a mild solution of the initial value problem* $\frac{d}{dt}g = \beta'(f)F$*,* $g(0) = \beta(f^0)$*; note that this is stronger than (20)). As a consequence, for all* $\psi \in L^\infty(\Omega)$*, we have that* $\int_\Omega \beta(f)\psi\,d\mu(x)$ *is absolutely continuous on* $[0,T)$ *and*

$$\frac{d}{dt}\int_\Omega \beta(f)\psi\,d\mu(x) = \int_\Omega \beta'(f)F\psi\,d\mu(x).$$

As the concept of being "piecewise" something might differ slightly from place to place, we state ours here:

Definition 2.18. Take an interval $I \subset \mathbb{R}$. A function $h : I \to \mathbb{R}$ is said to be *piecewise continuous* if there is a finite set of points $x_1 < x_2 < \cdots < x_N \in I$ such that

- h is continuous on $I \setminus \{x_1, \dots, x_N\}$,
- both $\lim_{x\to x_i^+} h(x)$ and $\lim_{x\to x_i^-} h(x)$ exist and are finite for every $i \in 1,\dots,N$,
- and h has a finite limit at any endpoint of I which belongs to I.

Note that the value of h at the x_i or at the endpoints of the intervals plays no role in the definition, so it makes sense to speak about piecewise continuous functions which are defined on all of I except for a finite number of points.

A function $h : I \to \mathbb{R}$ is *piecewise* $\mathcal{C}^1$ if it is continuous at every point, differentiable at all but a finite number of points, and h' is piecewise continuous.

To prove the above theorem we will need to take several steps. First, let us prove that (20) holds, in the sense of distributions, for a function β which is piecewise $\mathcal{C}^1$, bounded and with a bounded derivative. Clearly it is enough to prove it when β' has only one point of discontinuity.

The first problem is to define what the product $\beta'(f)F$ means: there is a problem because there can be a point a where $\beta'(a)$ is not defined, and it might happen that $f(t,x) = a$ for all (t,x). We will see that, even if this happens, the product $\beta'(f)F$ is always well defined; in our example, observe that F would be zero almost everywhere and thus there is no problem in defining the product. This is a version of Sard's theorem: if f is constant on a set so big that it is a problem to define $\beta'(f)$ there, then F is almost everywhere zero on that set and there is no problem with the definition of the product.

So take any $\beta : \mathbb{R} \to \mathbb{R}$ which is piecewise $\mathcal{C}^1$, is bounded and has a bounded derivative which only has one point of discontinuity $a \in \mathbb{R}$ (where β' is undefined). Approximate β by functions $\beta_n \in \mathcal{C}^{1,b}$ so that

- $\beta_n \to \beta$ uniformly,
- $\beta_n'(s) \to \beta'(s)$ for any $s \neq a$,
- $\beta_n'(a) \to \omega$ for some $\omega \in \mathbb{R}$.

Note that we can find such an approximation for any ω we like. This means that $\beta_n' \to \gamma$ pointwise, where

$$\gamma(s) := \begin{cases} \beta'(s) & \text{for } s \neq a \\ \omega & \text{for } s = a \end{cases}$$

For β_n, we know that (20) holds: for any $\varphi \in \mathcal{C}_c^\infty((0,T)\times\Omega)$,

$$-\int_0^T \int_\Omega \beta_n(f(s,x))\partial_t\varphi(s,x)\,d\mu(x)\,ds = \int_0^T \int_\Omega \beta_n'(f(s,x))F(s,x)\varphi(s,x)\,d\mu(x)\,ds.$$

Thanks to the convergence of β_n we can pass to the limit and say that

$$-\int_0^T \int_\Omega \beta(f(s,x))\partial_t\varphi(s,x)\,d\mu(x)\,ds = \int_0^T \int_\Omega \gamma(f(s,x))F(s,x)\varphi(s,x)\,d\mu(x)\,ds. \tag{23}$$

But we can obtain the same with a different value of ω. As this only affects the right hand side, we deduce that its value is independent of ω, and thus

$$\int_0^T \int_\Omega \chi_{f=a}(s,x)F(s,x)\varphi(s,x)\,d\mu(x)\,ds = 0.$$

(Here, $\chi_{f=a}$ represents the characteristic function of the set $\{(s,x) \in (0,T)\times\Omega \mid f(s,x) = a\}$). As φ is arbitrary, this means that

$$\chi_{f=a}F \text{ is zero almost everywhere.} \tag{24}$$

As we can change our β, the latter affirmation must be true for any $a \in \mathbb{R}$. This enables us to state the following:

Lemma 2.19. *Let $\beta : \mathbb{R} \to \mathbb{R}$ be a piecewise $\mathcal{C}^1$ function which is bounded and has a bounded derivative. Then the product $\beta'(f)F$ is defined almost everywhere on $(0,T) \times \Omega$ and is an integrable function which is independent of the values of β' at its points of discontinuity.*

With this, we can substitute γ in (23) by $\beta'(f)$ (as we know the resulting product does not depend on the value ω at the point of discontinuity) and obtain the following:

Proposition 2.20. *Let $\beta : \mathbb{R} \to \mathbb{R}$ be a piecewise $\mathcal{C}^1$ function which is bounded and has a bounded derivative. Then*

$$\partial_t \beta(f) = \beta'(f)F \quad \text{in } \mathcal{D}'((0,T) \times \Omega). \tag{25}$$

In fact, the above holds in a stronger sense:

$$\beta(f(t)) = \beta(f^0) + \int_0^t \beta'(f(s))F(s)\, ds \quad \text{for all } t \in (0,T).$$

Proof. It only remains to prove that the stronger version holds. As $\beta'(f)F$ is locally integrable, we already know that $\beta(f)$ is a weak solution to the initial value problem $\frac{d}{dt}g = \beta'(t)F(t)$ (see remark 2.9); as all our definitions are equivalent, it is also a mild solution to this initial value problem, which proves the result. $\square$

Proof of theorem 2.17. We have to prove that the previous proposition is still true when $\beta : \mathbb{R} \to \mathbb{R}$ is a piecewise $\mathcal{C}^1$ function such that both $\beta(f^0)$ and $\beta'(f)F$ are integrable (in Ω and $(0,T) \times \Omega$, respectively; remember that we always use the measure μ on Ω).

If β is such a function, we can approximate it by functions β_n which are piecewise $\mathcal{C}^1$ and of compact support such that

- $|\beta_n| \leq |\beta|$,
- β_n is equal to β on $[-n, n]$,
- $|\beta_n'| \leq 1$ on $\mathbb{R} \setminus [-n, n]$.

We know that the proposition holds for β_n:

$$\beta_n(f(t)) = \beta_n(f^0) + \int_0^t \beta_n'(f(s))F(s)\,ds \quad \text{for all } t \in (0,T). \qquad (26)$$

By the monotone convergence theorem (as $\beta(f^0)$ is integrable), the first term in the right hand side converges to $\beta(f^0)$ in $L^1(\Omega)$. Let us see that the second term also converges in $L^1(\Omega)$: for an integer n, consider the set

$$E_n := \{(t,x) \in (0,T) \times \Omega \mid |f(t,x)| \geq n\}.$$

The $dt \otimes \mu$-measure of the set E_n tends to 0 as $n \to \infty$, as f is integrable on $(0,T) \times \Omega$. Also, we have that

$$\begin{aligned}
\int_\Omega \int_0^t |\beta_n'(f(s)) - \beta'(f(s))|\,|F(s)|\ ds\,d\mu &= \int_{E_n} |\beta_n'(f(s)) - \beta'(f(s))|\,|F(s)|\ ds\,d\mu \\
&\leq \int_{E_n} |\beta_n'(f(s))|\,|F(s)|\ ds\,d\mu + \int_{E_n} |\beta'(f(s))|\,|F(s)|\ ds\,d\mu \\
&\leq \int_{E_n} |F(s)|\ ds\,d\mu + \int_{E_n} |\beta'(f(s))|\,|F(s)|\ ds\,d\mu.
\end{aligned}$$

The previous expression tends to 0 as $n \to \infty$. Hence,

$$\int_0^t \beta_n'(f(s))F(s)\,ds \to \int_0^t \beta'(f(s))F(s)\,ds \quad \text{in } L^1(\Omega).$$

Then, we deduce that the left hand side of (26) converges in $L^1(\Omega)$ as $n \to \infty$; as it converges pointwise a.e., we know it converges in $L^1(\Omega)$ to $\beta(f(t))$. Passing to the limit in (26) we finally obtain that

$$\beta(f(t)) = \beta(f^0) + \int_0^t \beta'(f(s))F(s)\,ds \quad \text{for all } t \in (0,T).$$

This finishes the proof. Note that the second part of theorem 2.17 follows from the first one, as we have proved that $\beta(f)$ is a mild solution to the equation $\frac{d}{dt}g = \beta'(f)F$, and hence also a solution in the sense of moments. □

3 About this text

This document has been written by José Alfredo Cañizo, based on notes by Stéphane Mischler. For comments or suggestions write to `ozarfreo@yahoo.com`. The latest version should be at `http://www.ugr.es/~ozarfreo/tex` .

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